## Chapter 15

## BUCKLING OF THE STRAIGHT BARS

### 15.1 GENERALS

Any time a construction element is subjected to compression (centric or eccentric), the possibility of loss of stability exists. For linear members (bars) this phenomenon is called buckling, while for plane members (plates) it is called local buckling. The behavior of a compressed bar is a very complex and interesting subject, in structural design. The study of buckling introduces some important simplifying hypothesis:

- the material is considered to be perfectly elastic
- the compressive load is axially applied
- the bar axis is perfectly straight

These hypotheses transform the bar into an ideal member, without imperfections. In reality, in a construction member we meet geometrical and structural imperfections, which are inevitable. These imperfections make the difference between the real bar and the ideal bar (the bar without imperfections)

### 15.1.1 The divergence of equilibrium

The physical model which best approximates the real phenomenon of instability is the one called by Dutheil: "Divergence of Equilibrium", which works with real bars. Let's consider a steel bar, affected by a geometrical imperfection, an initial curvature $\mathrm{w}_{0}$ (Fig.15.1).



Fig.15.1
For the undistorted position of the bar $\left(\mathrm{w}_{0}=0\right)$ we define the first order moment: $\mathrm{M}^{\mathrm{L}}=\mathrm{F} \cdot \mathrm{w}_{2}$. For the real bar with initial curvature, it is defined the second order moment, frequently used in the study of the bars stability: $\mathrm{M}^{\mathrm{II}}=\mathrm{F} \cdot \mathrm{w}$

The axial force: $\mathrm{N}=\mathrm{F}$



Fig.15.2
The compressive force F grows continually. The moment $\mathrm{M}^{\mathrm{II}}$ depends on F but also by the transversal deformation of the bar. That's why the
moment $\mathrm{M}^{\mathrm{II}}$ grows faster than the axial force and the equilibrium curve: axial force F and maximum deformation w (Fig.15.2), is no longer linear.

If the force F grows continually the normal stress $\sigma$ reaches the yield limit $\sigma_{\mathrm{c}}$, corresponding to the point C in the above graphic (Fig.15.2). Until the point D from the curve, the exterior moment $\mathrm{M}^{\mathrm{I}}=\mathrm{F} \cdot \mathrm{w}$ is in equilibrium with the interior moment $\mathrm{M}=\int_{A} \sigma \cdot z \cdot d A$

## In point $D$ :

- the exterior moment $\mathrm{M}^{\mathrm{II}}$ grows unlimited ( F grows very much)
- the interior moment M is limited as value, because: the lever arm is limited by the height of the cross section, and $\sigma$ is limited by the yield limit $\sigma_{C}$
- in this point these 2 moments are in divergence and the bar will buckle


### 15.1.2 The bifurcation of equilibrium

This model works with ideal bars, without imperfections. Talking about equilibrium, we consider a ball, in 3 situations:

- on a concave surface:

- on a plane surface:

- on a convex surface:


Using a disruptive (disturbing) force, the ball is no longer into its initial position. Removing the force we can observe what's happening with the ball:

- for the concave surface: a stable equilibrium (the ball is again in the initial position)
- for the plane surface: a neutral equilibrium
- for the convex surface: an unstable equilibrium

From energetically point of view, the equilibrium nature can be studied using the variation of the potential energy of the system $\Delta \pi$ :

For the concave surface: $\quad \Delta \Pi>0$
For the plane surface: $\quad \Delta \Pi=0$
For the convex surface: $\quad \Delta \Pi<0$
In the bifurcation of equilibrium, it is used the situation of neutral equilibrium, with $\Delta \Pi=0$.



Fig. 15.4
In point $B$, the equilibrium curve is bifurcated.

### 15.2 THE CALCULATION OF THE CRITICAL FORCE OF BIFURCATION (CRITICAL BUCKLING FORCE)

### 15.2.1 The static method

Consider a pin-ended column (Fig.15.5) axially compressed (centric compression) by the force F. When this load is increased, to a value of the load, the column becomes unstable, transverse deflection occurs and the result is the collapse. If the column is slender, buckling occurs at a stress below the yield limit and this critical stress is not related to the strength of material. This phenomenon is called elastic instability or buckling.

The transverse deflection, or buckling, occurs in the weakest plane of the cross section, perpendicular to the axis having the minimum moment of inertia. It is written that the external moment M, written about the centroid of the cross section: $\mathrm{M}=\mathrm{F} \cdot \mathrm{w}$, is equal to the internal moment expressed in terms of the curvature of the deflected shape: $\frac{d^{2} w}{d x^{2}}=-\frac{M}{E I}$


Fig.15.5

So: $\quad \frac{d^{2} w_{M}}{d x^{2}}=-\frac{M}{E I}=-\frac{F}{E I} \cdot w$
But the differential equation of the deflected axis must consider also the effect of the shear force $V$ :
$\bar{A}=\frac{A}{X}:$ transformed area
$\frac{\mathrm{dw}_{\mathrm{V}}}{\mathrm{dx}}=\frac{\mathrm{V}}{\mathrm{G} \bar{A}}=\frac{X \cdot V}{\mathrm{GA}}$
$\frac{\mathrm{d}^{2} \mathrm{w}_{V}}{\mathrm{dx}^{2}}=\frac{X \cdot V^{\prime}}{\mathrm{GA}}=\frac{X \cdot \mathrm{M}^{\prime \prime}}{\mathrm{GA}}$
But: $\mathrm{w}=w_{M}+w_{V}$
The complete differential equation is:
$\frac{d^{2} w}{d x^{2}}=\frac{d^{2} w_{M}}{d x^{2}}+\frac{d^{2} w_{V}}{d x^{2}}$
$\frac{\mathrm{d}^{2} \mathrm{w}}{\mathrm{dx}^{2}}=\frac{-\mathrm{M}}{\mathrm{EI}}+\frac{X \cdot \mathrm{M}^{\prime \prime}}{\mathrm{GA}}$
$\mathrm{M}=\mathrm{F} \cdot \mathrm{w}$ and $\mathrm{M} "=\mathrm{F} \cdot \frac{\mathrm{d}^{2} \mathrm{w}}{\mathrm{dx}^{2}}$
$\frac{\mathrm{d}^{2} \mathrm{w}}{\mathrm{dx}^{2}}-\frac{X \cdot \mathrm{~F}}{\mathrm{GA}} \cdot \frac{\mathrm{d}^{2} \mathrm{w}}{\mathrm{dx}^{2}}+\frac{\mathrm{F}}{\mathrm{EI}} \cdot \mathrm{w}=0$
$\frac{\mathrm{d}^{2} \mathrm{w}}{\mathrm{dx}^{2}}\left(1-\frac{X \cdot \mathrm{~F}}{\mathrm{GA}}\right)+\frac{\mathrm{F}}{\mathrm{EI}} \cdot \mathrm{w}=0$
It is noted: $\frac{\mathrm{F}}{\mathrm{EI}\left(1-\frac{X \cdot \mathrm{~F}}{\mathrm{GA}}\right)}=\mathrm{k}^{2}$
$\frac{\mathrm{d}^{2} \mathrm{w}}{\mathrm{dx}^{2}}+\mathrm{k}^{2} \cdot \mathrm{w}=0$
The general solution of equation (4) is:
$\mathrm{w}=\mathrm{A} \sin \mathrm{kx}+\mathrm{B} \cos \mathrm{kx}$
If we write the boundary conditions:

- $\quad$ for $x=0: w=0=>B=0$
- for $x=1: w=0 \Rightarrow A \sin k l=0$

$$
\begin{align*}
& \mathrm{A} \neq 0=>\sin \mathrm{kl}=0=>\mathrm{kl}=\mathrm{n} \pi  \tag{5}\\
& \mathrm{k}^{2} \mathrm{l}^{2}=\mathrm{n}^{2} \pi^{2}, \text { replacing }(3)=>\frac{\mathrm{F}}{\mathrm{EI}\left(1-\frac{X \cdot \mathrm{~F}}{\mathrm{GA}}\right)}=\frac{\mathrm{n}^{2} \pi^{2}}{\mathrm{l}^{2}} \\
& \frac{\mathrm{~F}}{1-\frac{X \cdot \mathrm{~F}}{\mathrm{GA}}}=\frac{\mathrm{n}^{2} \pi^{2} \mathrm{EI}}{\mathrm{l}^{2}}
\end{align*}
$$

Noting: $\frac{\mathrm{n}^{2} \pi^{2} \mathrm{EI}}{\mathrm{l}^{2}}=F_{c r}^{M}$ : the critical force of buckling only from bending moment M

$$
\begin{equation*}
\frac{\mathrm{F}}{1-\frac{X \cdot \mathrm{~F}}{\mathrm{GA}}}=F_{c r}^{M} \rightarrow F\left(1+\frac{X \cdot \mathrm{~F}_{c r}^{M}}{\mathrm{GA}}\right)=F_{c r}^{M} \rightarrow F_{c r}=\frac{F_{c r}^{M}}{1+\frac{X \cdot \mathrm{~F}_{c r}^{M}}{\mathrm{GA}}} \tag{6}
\end{equation*}
$$

For bars made from one piece (rectangular, circular, hollow pipe, a rolled profile) the influence of the shear force V is neglected, so the term with $X$ will be zero. Relation (6) will become:

$$
\begin{equation*}
\mathrm{F}_{\mathrm{cr}}=F_{c r}^{M}=\frac{\mathrm{n}^{2} \pi^{2} \mathrm{EI}}{\mathrm{l}^{2}} \tag{7}
\end{equation*}
$$

If the compressed bar is made from many profiles (Fig.15.6) connected between them (composed bars), the influence of the shear force V is important.


Fig. 15.6

Considering only the term from M , the relation (7) provides us a critical force for each value of $n$ :

For $\mathrm{n}=1 \quad$ For $\mathrm{n}=2$


We are interested only by the minimum critical force of buckling, which corresponds to $\mathrm{n}=1$. So, for a pin-ended bar, the critical force is:

$$
\mathrm{F}_{\mathrm{cr}}=\frac{\pi^{2} \mathrm{EI}}{1^{2}}
$$

The above relation was first written by Leonhard Euler, in 1744.
$\mathrm{F}_{\mathrm{cr}}$ represents Euler's critical force of buckling.

### 15.2.2 The energetically method

As we saw, for a neutral equilibrium, the variation of the potential energy $\Delta \Pi=0$.

Let's consider a simple supported beam and a level of reference $H$ (Fig.15.7).

$$
\Pi=0 \quad \Pi=\mathrm{P} \cdot \mathrm{H} \quad \Pi=\mathrm{P}(\mathrm{H}-\mathrm{w})+\mathrm{U}_{\mathrm{d}}
$$






Fig. 15.7
$\mathrm{U}_{\mathrm{d}}$ : the strain energy produced in bar by the internal stresses $\left(\mathrm{N}, \mathrm{M}, \mathrm{V}, \mathrm{M}_{\mathrm{t}}\right)$
For $H=0 \Rightarrow \quad \pi=U_{d}-P \cdot w$
Nothing $\quad \mathrm{L}=\mathrm{P} \cdot \mathrm{w} \rightarrow$ the mechanical work of the exterior forces

$$
w \rightarrow \text { vertical displacement of the force } P
$$

Replacing L: $\quad \pi=U_{d}-L$
For a neutral equilibrium $\Delta \pi=0$
$\Delta \mathrm{U}_{\mathrm{d}}=\Delta \mathrm{L} \rightarrow$ Based on this equality, between the variation of the strain energy $\Delta U_{d}$ and the variation of the mechanical work $\Delta L$, the critical force of buckling $\mathrm{F}_{\text {cr }}$ may be calculated

Let's consider a simple supported column, axially compressed by a force F (Fig.15.8).


Fig. 15.8

The strain energy:

$$
\begin{aligned}
& \mathrm{U}_{\mathrm{d}(1)}=\frac{1}{2} \int_{\mathrm{O}}^{\mathrm{l}} \frac{\mathrm{~N}^{2}}{\mathrm{EA}} \mathrm{dx} \\
& \mathrm{U}_{\mathrm{d}(2)}=\frac{1}{2} \int_{\mathrm{O}}^{1} \frac{\mathrm{~N}^{2}}{\mathrm{EA}} \mathrm{dx}+\frac{1}{2} \int_{\mathrm{O}}^{1 \mathrm{M}^{2}} \frac{\mathrm{EI}}{\mathrm{EI}} \mathrm{dx}
\end{aligned}
$$

$$
\Delta \mathrm{U}_{\mathrm{d}}=\mathrm{U}_{\mathrm{d}(2)}-\mathrm{U}_{\mathrm{d}(1)}=\frac{1}{2} \int_{\mathrm{o}}^{1} \frac{\mathrm{M}^{2}}{\mathrm{EI}} \mathrm{dx}
$$

$$
\Delta \mathrm{U}_{\mathrm{d}}=\frac{1}{2} \int_{\mathrm{o}}^{1} \frac{\mathrm{M}^{2}}{\mathrm{EI}} \mathrm{dx}
$$

The mechanical work:
$\mathrm{L}_{(1)}=0$
$\mathrm{L}_{(2)}=\mathrm{F} \cdot \mathrm{u}_{1} \quad \Rightarrow \quad \Delta \mathrm{~L}=\mathrm{F} \cdot \mathrm{u}_{1}$
But, from the differential equation of the strained axis:
$w^{\prime \prime}=\frac{d^{2} w}{d^{2}}=-\frac{M}{E I}=>M=-w^{\prime \prime} \cdot E I$

And : $\quad \Delta \mathrm{U}_{\mathrm{d}}=\frac{1}{2} \int_{\mathrm{o}}^{\mathrm{l}} \mathrm{EI}\left(\mathrm{w}^{\prime \prime}\right)^{2} \mathrm{dx}$
(a)

To explain $\Delta \mathrm{L}$ (Fig.15.9) we must express the vertical displacement $\mathrm{u}_{1}$.


Fig. 15.9
On the entire length of the column:
$u_{1}=\int_{0}^{1} d u=\frac{1}{2} \int_{0}^{1}\left(w^{\prime}\right)^{2} d x$
So : $\Delta \mathrm{L}=\frac{\mathrm{F}}{2} \int_{0}^{1}\left(\mathrm{w}^{\prime}\right)^{2} \mathrm{dx}$
From $(a)=(b)$, the critical force of buckling $\mathrm{F}_{\text {cr }}$ will be:

$$
\boldsymbol{F}_{c r}=\frac{\int_{o}^{l} E I\left(w^{\prime \prime}\right)^{2} d x}{\int_{o}^{l}\left(w^{\prime}\right)^{2} d x}
$$

In the formula of $\mathrm{F}_{\mathrm{cr}}$ the expression of $w$ is unknown. $w$ is the analytical expression of the deflected shape of the bar, in the moment of reaching the neutral equilibrium.

In calculations $\mathrm{w}=\mathrm{f}(\mathrm{x})$, chosen in order to respect the boundary conditions.

Example:


$$
\mathrm{w}(\mathrm{x})=\mathrm{A} \sin \frac{\pi \mathrm{x}}{1}
$$

Boundary conditions:

- for $\mathrm{x}=0 \rightarrow \mathrm{w}=0$
- for $\mathrm{x}=1 \rightarrow \mathrm{w}=0$

Both conditions are verified.

$$
\begin{aligned}
& \mathrm{w}^{\prime}=\mathrm{A} \frac{\pi}{1} \cos \frac{\pi \mathrm{x}}{1} \\
& \mathrm{w}^{\prime \prime}=-\mathrm{A} \frac{\pi^{2}}{1^{2}} \sin \frac{\pi x}{1}
\end{aligned}
$$



### 15.2.3 Generalized Euler's formula

$\mathbf{F}_{\mathbf{c r}}=\frac{\mathbf{\pi}^{\mathbf{2}} \mathbf{E I}_{\mathbf{m i n}}}{\mathbf{l}_{\mathrm{f}}^{2}} \quad E I_{\text {min }}$ : the rigidity of the bar about the min. axis of inertia

$\mathrm{l}_{\mathrm{f}}$ : is the length of buckling (Fig.15.10)

## The critical buckling stress:

$\sigma_{\text {cr }}=\frac{\mathrm{F}_{\mathrm{cr}}}{\mathrm{A}}=\frac{\pi^{2} \mathrm{EI}_{\text {min }}}{\mathrm{l}_{\mathrm{f}}^{2} \mathrm{~A}}$
$\mathrm{i}_{\text {min }}=\sqrt{\frac{\mathrm{I}_{\text {min }}}{\mathrm{A}}}$
$\sigma_{\mathrm{cr}}=\frac{\pi^{2} \mathrm{E} \cdot \cdot_{\min }^{2}}{1_{\mathrm{f}}^{2}}=\frac{\pi^{2} \mathrm{E}}{\left(\frac{1_{\mathrm{f}}}{i_{\text {min }}}\right)^{2}} \rightarrow \sigma_{\mathrm{cr}}=\frac{\boldsymbol{\pi}^{2} \mathbf{E}}{\lambda^{2}}$
$\lambda=\frac{\mathrm{l}_{\mathrm{f}}}{\mathrm{i}_{\text {min }}}:$ is the slenderness ratio $\rightarrow$ characteristic of the bar in what concern its stability, depending on the length and the supports of the bar, the shape and the magnitude of its cross section and the type of steel (through its yield limit).

For an ideal elasto-plastic material (example steel) the relation of $\sigma_{\text {cr }}$ $=\frac{\pi^{2} \mathrm{E}}{\lambda^{2}} \quad$ can be represented in a reference system $\sigma_{\mathrm{cr}}-\lambda$, as a cubic hyperbola, called Euler's hyperbola (Fig.15.11):


Fig.15.11

It can be observed that the critical stress at buckling isn't a constant of the material, because it depends on the slenderness $\lambda$. The hyperbola is limited at the value of the yield limit $\sigma_{\mathrm{c}}$, because the critical stress was determined considering an ideal elasto-plastic material. In what concern the slenderness $\lambda$, the hyperbola is limited to $\lambda_{\lim }$ :
$\sigma_{\mathrm{cr}}=\frac{\pi^{2} \mathrm{E}}{\lambda^{2}}=\sigma_{\mathrm{c}}=>\lambda_{\lim }=\pi \sqrt{\frac{\mathrm{E}}{\sigma_{\mathrm{c}}}}$
Ex.: for common steel OL37 (S235): $\mathrm{E}=2,1 \times 10^{6} \mathrm{daN} / \mathrm{cm}^{2}, \sigma_{\mathrm{c}}=2400$ daN/cm ${ }^{2}$
$\lambda_{\lim }=\pi \sqrt{\frac{2,1 \times 10^{6}}{2400}} \cong 93 \rightarrow \operatorname{curve}(1)$
For $\lambda>\lambda_{\text {lim }}$, $\sigma_{\text {cr }}$ has smaller values as $\sigma_{\mathrm{c}}$, the ideal bar has an elastic behavior and we discuss about a problem of stability; the bar fails by buckling.

For $\lambda<\lambda_{\text {lim }}, \sigma_{\mathrm{cr}}$ has a constant value $\sigma_{\mathrm{cr}}=\sigma_{\mathrm{c}}$, and now it is a problem of strength verification in plastic domain; the bar fails by yielding.

The above hyperbola is a theoretical one, because it neglects the imperfections of the real bar.

Engesser used a tangent modulus of elasticity Et (taking into account the remnant stresses from the rolled profiles to the irregular cooling after rolling), used only between the limit of proportionality $\sigma_{\mathrm{p}}$ and yield limit $\sigma_{\mathrm{c}}$. So:
$\sigma_{\mathrm{cr}}=\frac{\pi^{2} \mathrm{E}_{\mathrm{t}}}{\lambda_{\mathrm{p}}^{2}}=\sigma_{\mathrm{p}} \Rightarrow \boldsymbol{\lambda}_{\mathrm{p}}=\pi \sqrt{\frac{\mathrm{E}_{\mathrm{t}}}{\sigma_{\mathrm{p}}}} \rightarrow$ curve (2)
Using the curves (1) and (2), the current standard for calculation the steel compressed elements works with 3 buckling curves $\mathrm{A}, \mathrm{B}$ and C for each type of material (defined by the yield strength $\mathrm{R}_{\mathrm{c}}$ ), for different type of cross sections (Fig.15.12).


Fig. 15.12

### 15.3. DESIGN OF AXIALLY COMPRESSED BARS USING THE METHOD OF COEFFICIENTS OF BUCKLING $\varphi$

The normal stress for centric compression calculated from stability condition:

$$
\sigma=\frac{\mathrm{N}^{\mathrm{d}}}{\varphi_{\min } \mathrm{A}} \leq \mathrm{R}
$$

$\mathrm{N}^{\mathrm{d}}$ : is the design compressive force
$\mathrm{A}=\mathrm{A}_{\text {gross }}:$ the gross area of the cross section
$\varphi_{\text {min }}$ : the minimum coefficient of buckling

- compute the length of buckling $l_{f}$
- compute $\mathrm{i}_{\text {min }}=\sqrt{\frac{\mathrm{I}_{\text {min }}}{\mathrm{A}}} \Rightarrow \lambda=\frac{\mathrm{l}_{\mathrm{f}}}{\mathrm{i}_{\text {min }}}$
- from tables, function the shape of the cross section we include it into a buckling curve $\mathrm{A}, \mathrm{B}$, or C
- from that table, function of $\lambda=>\varphi_{\min } \leq 1,0$

