Chapter14

CALCULATION OF SECTIONS IN ELASTO-PLASTIC DOMAIN

14.1 GENERALS

In all strength calculations made until now, it was used the **hypothesis of the linear elastic materia**l, where **Hooke's law** is valid until the **limit state** of the material **(yield limit).** The unit stresses should not exceed the design strength R.

But, for a **ductile material**, if the stress distribution isn't uniform, **the reaching of the yield limit** in a point doesn't produce the failure of the element or the construction. After the setting up of plastic deformations in a construction the strength capacity isn't consumed. To reach the limit state of a construction, the **load must be increased**.

To study an element beyond the limit of elasticity, the material is considered **isotropic** and **homogen**, it's **behavior** being shown (Fig. 14.1) in the **ideal elasto-plastic curve** (Prandtl's diagram) for ductile materials with yield plateau:

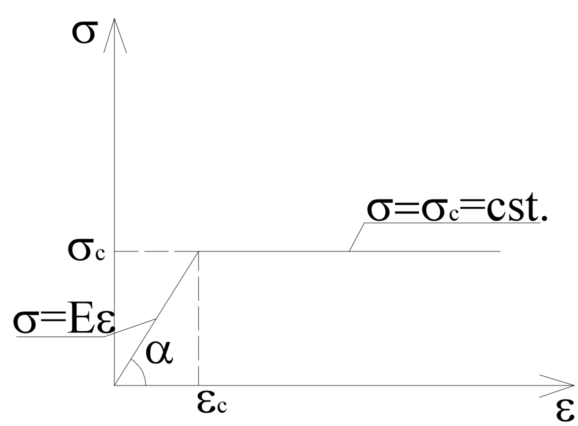


Fig. 14.1

The yield limit σ_c for common steel is:

- In Romanian standard for OL37: $R_c = 2400 daN/cm^2$
- In Eurocode 3 for S235: $f_y = 235 N/mm^2$

14.2 PURE BENDING IN ELASTO-PLASTIC DOMAIN

We consider a bar with a mono-symmetric cross section subjected to pure bending (Fig. 14.2).

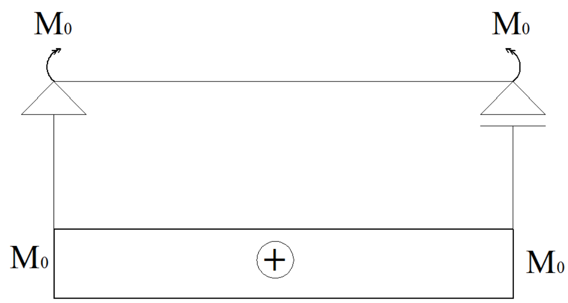


Fig. 14.2

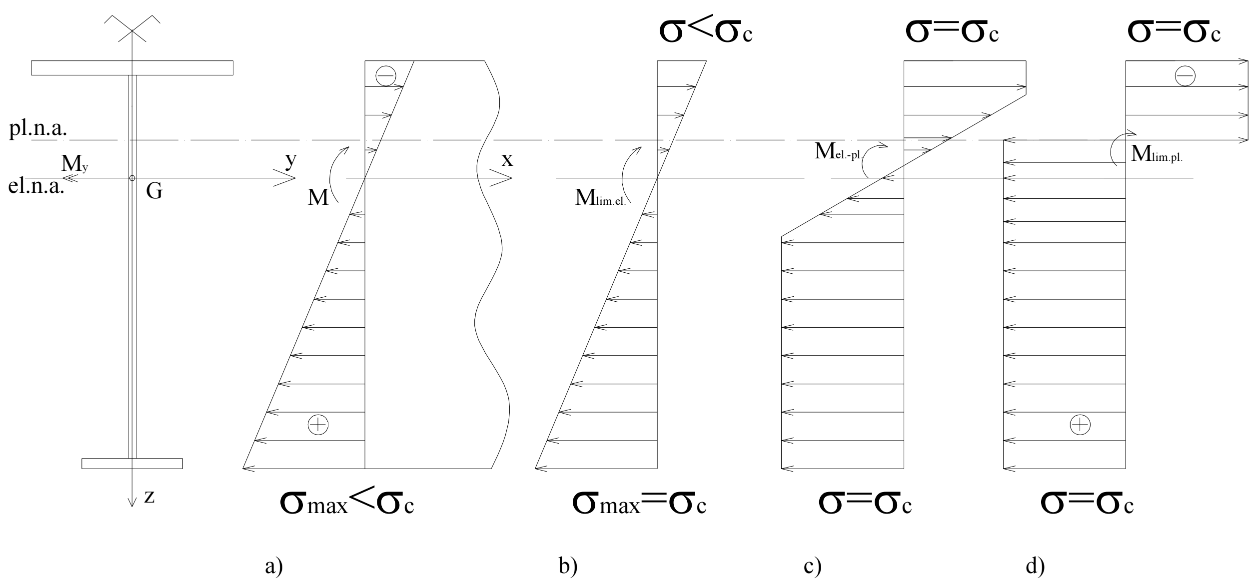


Fig. 14.3

In diagram (a) from Fig.14.3 the bending moment M_y produces a normal stress σ_x characteristic to a linear elastic material, the stress distribution being linear with zero value in the neutral axis. The maximum normal stress σ_{xmax} doesn't reach yet the yield limit σ_c.

$$(a): \sigma_x = \frac{M_y}{W_y} < \sigma_c$$

We continue to increase the load until in the most subjected fiber the normal stress reaches σ_c (diagram (b)). In this situation we compute the limit elastic moment $M_{lim.el.}$.

$$(b): \sigma_x = \frac{M_{lim.el.}}{W_y} = \sigma_x \Longrightarrow M_{lim.el.} = W_{yel} \cdot \sigma_c$$

$$W_{yel} = \frac{I_y}{z_{max}}$$

If the load continues to increase, the specific linear deformations ε are increased, respecting the hypothesis of the plane sections. In the plasticized zone ε exceed the yield specific deformation ε_c . According to Prandtl's diagram, if $\varepsilon > \varepsilon_c$ the normal stress σ remain constant and equal to σ_c. In the remained elastic zone Hooke's low is still valid σ being proportional to ε. The stress distribution from diagram (c) shows that in the extreme zones σ reach σ_c , these zones being completely plasticized, while a central zone is still in the elastic domain, with the characteristic linear variation.

If the load is increased, the elastic zone is reduced and the limit situation is the one when the entire cross section is plasticized. We say that in that zone it was produced a **plastic hinge** (d). In diagram (d) from Fig.14.3 we observe a different distribution of the normal stress, with rectangular blocks in all fibers, $\sigma = \sigma_c$. In this situation we compute the limit plastic moment $M_{lim.pl.}$ (Fig.14.4), still unknown because we observe that now the passing from the tensile section of area A_t to the compressed section of area A_c is made at an unknown level, corresponding to the plastic neutral axis.

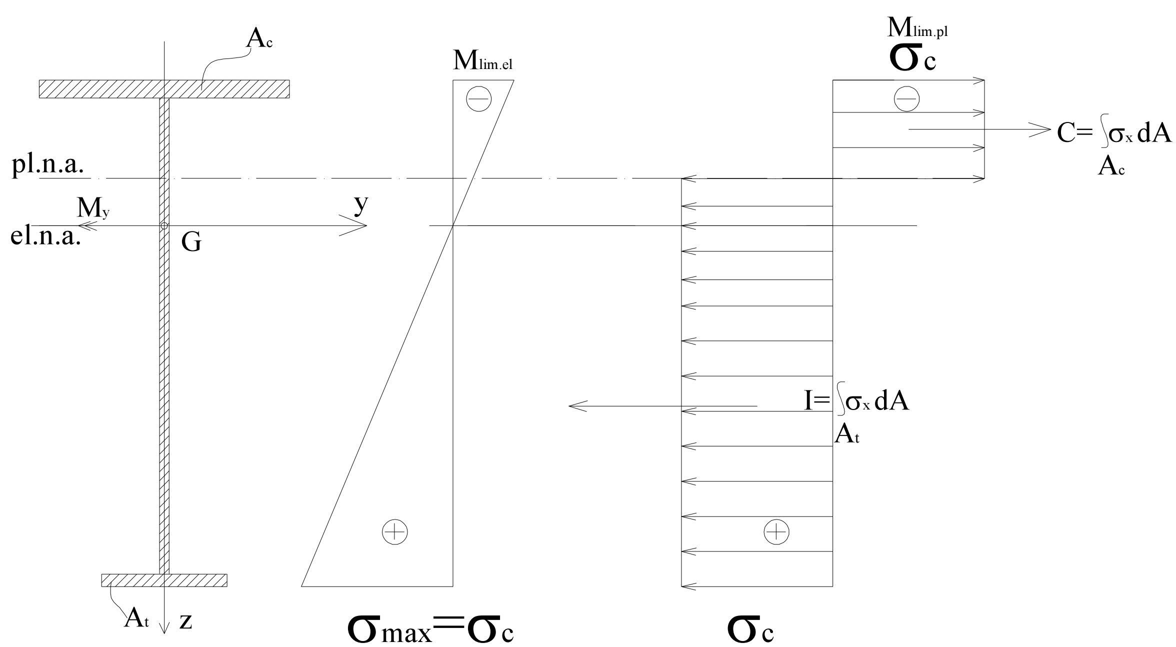


Fig.14.4

To determine the plastic neutral axis position, we write the axial force from a strength calculus N^{res} which must be equal to the one determined from static calculus $N^{res} = N^{st}$. But, from the bar loading the single distinct stress is the bending moment M_y , so: $N^{res} = N^{st} = 0$.

$$N^{res} = \int_A \sigma_x dA = \int_{A_t} \sigma_x dA - \int_{A_c} \sigma_x dA = \sigma_c \int_{A_t} dA - \sigma_c \int_{A_c} dA = \sigma_c(A_t - A_c) = 0 \rightarrow$$

$$\boxed{A_t = A_c}$$

This final relation shows that the **<u>plastic neutral axis divides the cross section into 2 equal areas (while the elastic neutral axis divides it in 2 parts of equal static moments)</u>**.

To compute the limit plastic moment, we write M_y from a strength calculation:

$$M_{lim.pl.} = \int_A (\sigma_x dA)z = \int_{A_t} (\sigma_x dA)z + \int_{A_c} (-\sigma_x dA)(-z) = \sigma_c \int_{A_t} z dA +$$

$$+\sigma_c \int_{A_c} z dA = \sigma_c (S_{y_t} + S_{y_c}) \rightarrow$$

$$\boxed{M_{lim.pl.} = W_{y_{pl}} \cdot \sigma_c}$$

$$\boxed{W_{y_{pl}} = S_{y_t} + S_{y_c}}$$

$W_{y_{pl}}$ - plastic strength modulus.

S_{y_t}, S_{y_c} – the static moment of the tensioned area respectively compressed area, written with respect to elastic neutral axis or plastic neutral axis

We can introduce a characteristic notion of the plastic calculus, called index of efficiency:

$$\boxed{i_e = \frac{W_{y_{pl}}}{W_{y_{el}}}}$$

For the rectangular cross section:

$$W_{y_{el}} = \frac{I_y}{z_{max}} = \frac{bh^3}{12} \frac{1}{\frac{h}{2}} = \frac{bh^2}{6}$$

$$W_{y_{pl}} = S_{y_t} + S_{y_c} = (\frac{bh}{2} \cdot \frac{h}{4})2 = \frac{bh^2}{4}$$

$$\boxed{i_e = \frac{W_{y_{pl}}}{W_{y_{el}}} = 1.5}$$

!!! For double symmetrical cross sections, the elastic neutral axis is identical to the plastic neutral axis

14.3 BENDING WITH SHEARING IN ELASTO-PLASTIC DOMAIN

Let's consider a simple supported beam loaded by a concentrate force in the middle span and having a rectangular cross section (Fig.14.5).

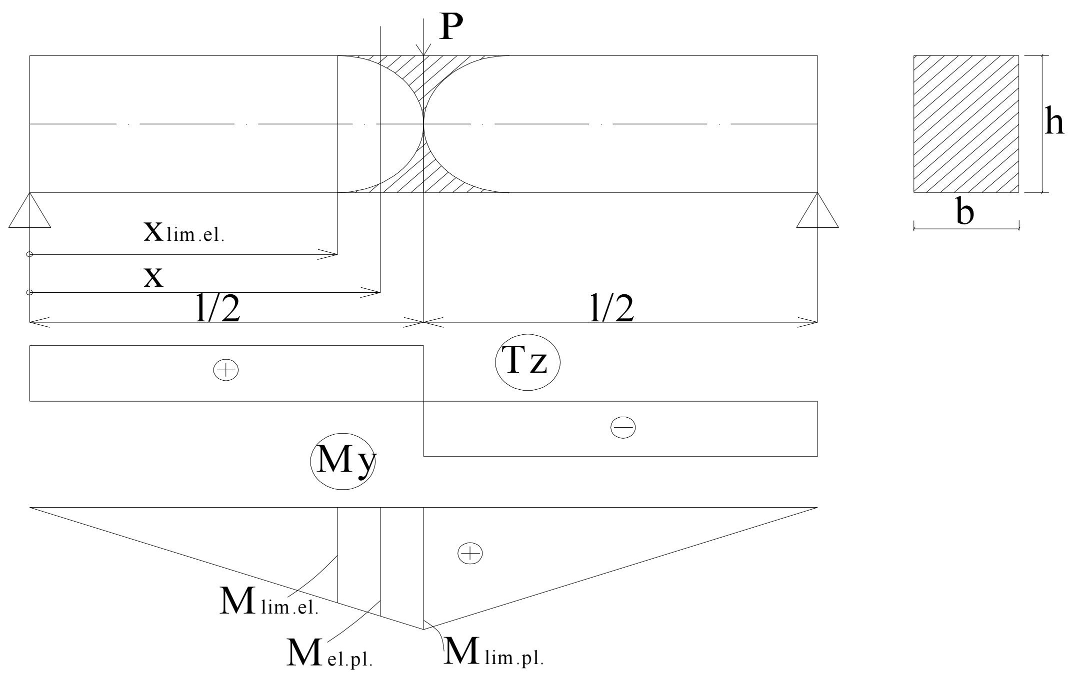


Fig.14.5

In case of pure bending the moment was constant along the bar and the plastic hinge was produced simultaneous in all the cross sections of the bar.

In case of bending with shearing the plastic hinge is produced in the section of maximum moment. The other cross sections are in elasto-plastic or elastic domain.

The section of maximum moment (in the middle span) is completely in plastic domain and $M_{max} = M_{lim.pl} = \frac{bh^2}{4} \cdot \sigma_c$, for $x = l/2$.

To a distance $x = x_{lim.el.}$, only the extreme fibers are in plastic domain, and:

$M = M_{lim.el}$

The variation of σ_x for the rectangular section of cross section $b \times h$, will be (Fig.14.6):

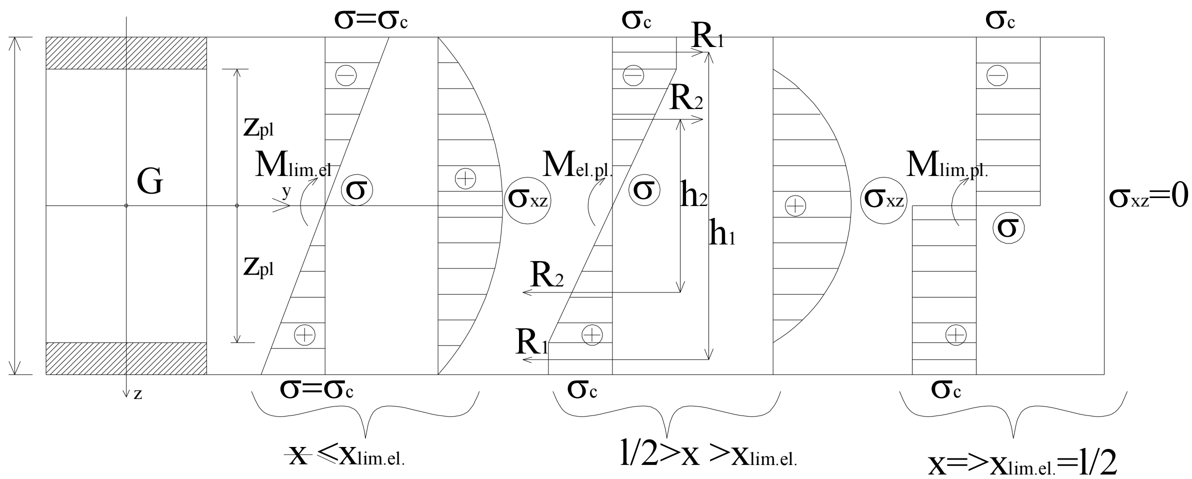


Fig.14.6

In an intermediary section (in elasto-plastic domain), where $x > x_{lim.el}$, the resultants of the normal stress σ_x are:

$$R_1 = b\left(\frac{h}{2} - z_{pl}\right)\cdot \sigma_c$$

$$R_2 = b \cdot \frac{1}{2} \cdot z_{pl} \cdot \sigma_c$$

The level arm of each couple R_1 and R_2:

$$h_1 = \left[z_{pl} + \frac{1}{2}\left(\frac{h}{2} - z_{pl}\right)\right] \cdot 2 = \frac{h}{2} + z_{pl}$$

$$h_2 = \frac{2}{3} \cdot z_{pl} \cdot 2 = \frac{4}{3} \cdot z_{pl}$$

The **elasto-plastic moment $M_{el.pl}$** , is:

$$M_{el.pl} = R_1 \cdot h_1 + R_2 \cdot h_2 = b\left(\frac{h}{2} - z_{pl}\right) \cdot \sigma_c \cdot \left(\frac{h}{2} + z_{pl}\right) + \frac{z_{pl}\cdot b}{2} \cdot \sigma_c \cdot \frac{4z_{pl}}{3} = b\cdot\sigma_c\left[\left(\frac{h^2}{4} - z_{pl}^2\right) + \frac{2z_{pl}^2}{3}\right] = b\cdot\sigma_c\left(\frac{h^2}{4} - \frac{z_{pl}^2}{3}\right) = \frac{bh^2}{4}\sigma_c\left(1 - \frac{4}{3}\frac{z_{pl}^2}{h^2}\right)$$

As: $W_{y.pl.} = \frac{bh^2}{4}$ and $M_{lim.pl.} = \frac{bh^2}{4}\sigma_c$ result:

$$M_{el.pl} = M_{lim.pl.}\left(1 - \frac{4}{3} \cdot \frac{z_{pl}^2}{h^2}\right)$$

So: $\frac{M_{el.pl.}}{M_{lim.pl.}} = 1 - \frac{4}{3} \cdot \frac{\mathbf{z_{pl}}^2}{\mathbf{h^2}}$ (a)

But, from the bending moment diagram:

$\frac{M_{el.pl.}}{M_{lim.pl.}} = \frac{x}{l/2} = \frac{2x}{l}$ (b)

From (a) and (b) => $\frac{\mathbf{2x}}{\mathbf{l}} = \mathbf{1} - \frac{\mathbf{4}}{\mathbf{3}} \frac{\mathbf{z_{pl}}^2}{\mathbf{h^2}}$ the equation of the second degree curves (parabola) which mark the limits of the plastic zone.

In what concern the shear stress τ:

- in section $x = \frac{l}{2}$ $\rightarrow$ $M_{max} = M_{lim.pl.}$

  $V = 0 \Rightarrow \tau = 0$

- in section $x \leq x_{lim.el}$ $\rightarrow$ $M = M_{lim.el.}$

  $V \neq 0 \Rightarrow \tau$ is calculated with Juravski's formula for elastic zone.

- in section $\frac{l}{2} > x > x_{lim.el.}$ $\rightarrow$ $M = M_{el.pl}$

  $V \neq 0 \Rightarrow \tau = 0$ in the plastic zones, $\tau \neq 0$ in the elastic zone.

14.4. BENDING WITH AXIAL FORCE IN PLASTIC DOMAIN

It is considered a rectangular cross section, entirely plasticized, by a bending moment $M_{lim.pl} > 0$ and a tensile force $N_{lim.pl} > 0$ (Fig.14.7):

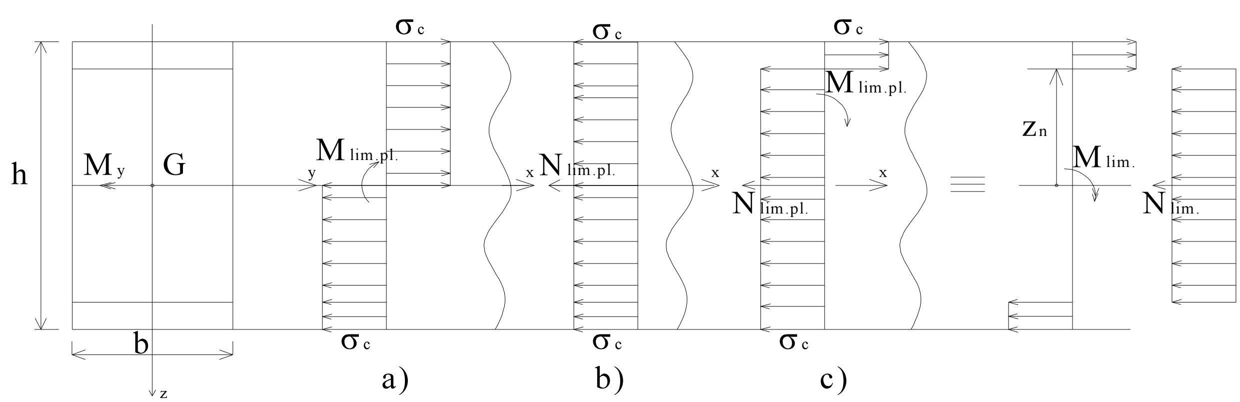


Fig.14.7

If the section is in plastic domain only from bending moment M, σ diagram is the one from Fig.14.7.a, and the corresponding moment is:

$$M_{lim.pl} = W_{ypl.} \cdot \sigma_c = \frac{bh^2}{4} \cdot \sigma_c = M_{pl.}$$

If the section is in plastic domain only from axial force N, σ diagram is the one from Fig.14.7.b, and the plastic axial force is:

$$N_{lim.pl.} = A \cdot \sigma_c = bh \cdot \sigma_c = N_{pl}$$

Considering the section in plastic domain from M and N, we obtain the diagram from Fig.14.7.c, for the normal stress σ. The corresponding stresses are M_{lim} and N_{lim}. The diagram from Fig.14.7.c is replaced by 2 diagrams: one corresponding to M_{lim} and the other to N_{lim} (Fig.14.8).

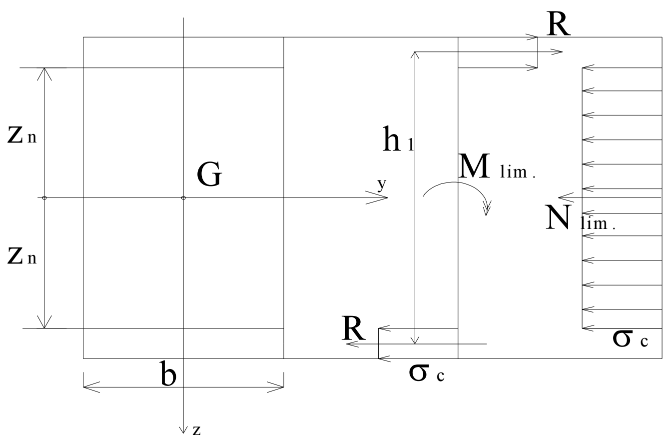


Fig.14.8

$$N_{lim} = A_n \cdot \sigma_c = 2z_n \cdot b \cdot \sigma_c \qquad (a)$$

$$M_{lim} = R \cdot h_l = (\frac{h}{2} - z_n) \cdot b \cdot \sigma_c (\frac{h}{2} + z_n) = b \cdot \sigma_c (\frac{h^2}{4} - z_n^2) \qquad (b)$$

From (a) → $z_n = \frac{N_{lim}}{2b\sigma_c}$ which is replaced in (b):

$$M_{lim} = b\sigma_c (\frac{h^2}{4} - \frac{N_{lim}^2}{4b^2\sigma_c^2}) = \frac{bh^2}{4} \cdot \sigma_c - \frac{N_{lim}^2}{4b\sigma_c} = M_{pl}(1 - \frac{4}{bh^2\sigma_c} * \frac{N_{lim}^2}{4b\sigma_c}) =$$

$$= M_{pl} (1 - \frac{N_{lim}^2}{b^2h^2\sigma_c}) = M_{pl} (1 - \frac{N_{lim}^2}{N_{pl}^2}) \rightarrow$$

$$\boxed{\frac{M_{lim}}{M_{pl}} + \left(\frac{N_{lim}}{N_{pl}}\right)^2 = 1}$$

Representing graphically this final interaction relation, 2 parabolas are obtained (Fig.14.9). If two values of N and M from a point of a cross section provide a point which is graphically represented inside these 2 parabolas, these 2 stresses N and M, can be overtaken by the cross section. If the point is on the boundary of this figure (one parabola) the stresses produces the complete plasticization of the cross section. If it is outside, N and M can be overtaken by cross section.

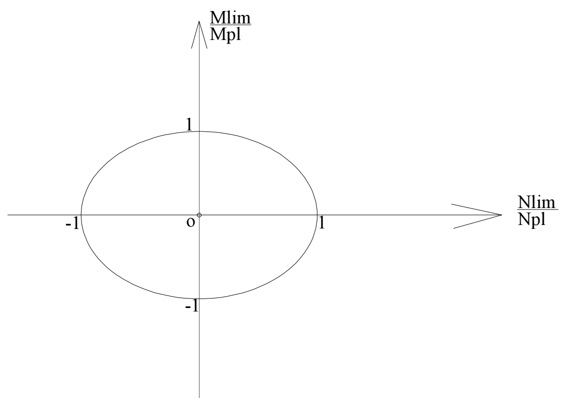


Fig.14.9

14.5 APPLICATION TO PLASTIC CALCULATION

For the following cross section subjected to pure bending by a positive bending moment M_y compute ($R_c = 2400 daN/cm^2$):

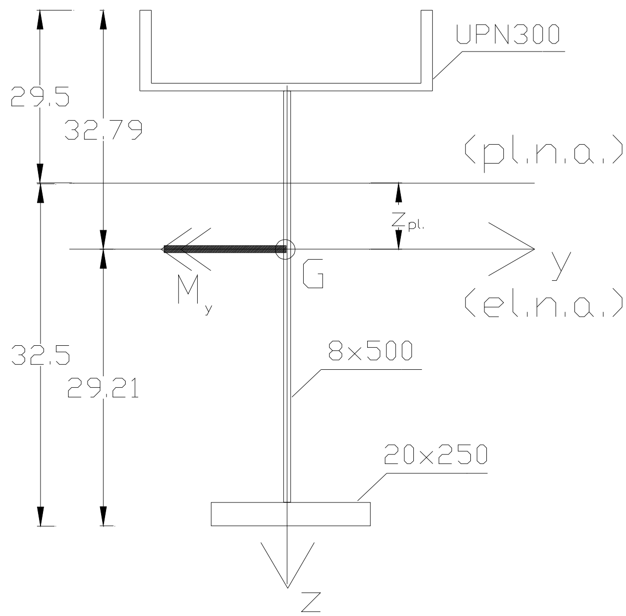
a. The limit elastic moment $M_{lim.el.}$
b. The limit plastic moment $M_{lim.pl.}$
c. The index of efficiency i_e

For the second moment of area $I_y = 87035\ cm^4$, the strength modulus about the elastic neutral axis *Gy* is:

$$W_{y\ el.} = \frac{87035}{32,79} = 2654,32 cm^3$$

The limit elastic moment $M_{lim.el.}$ is:

$M_{lim.el.} = W_{y\ el} \cdot R_c = 2654,32 \cdot 2400 \cdot 10^{-4} = 637$ kNm



The plastic neutral axis position results from the condition that the tensile area and the compressed area (with respect to pl.n.a) are equal.

The tensile area:

$A_t = 50 + (27{,}21 + z_{pl.}) \cdot 0{,}8$

$A_c = 58{,}8 + (22{,}79 - z_{pl.}) \cdot 0{,}8$

Equating $A_t = A_c \rightarrow z_{pl.} = 3{,}29$cm

The plastic strength modulus about the plastic neutral axis (pl.n.a) is:

$$\text{Pl.n.a.}\begin{cases} S_{yt} = 50 \cdot 31{,}5 + 30{,}5 \cdot 0{,}8 \cdot 15{,}25 = 1947{,}1 cm^3 \\ S_{yc} = 58{,}8 \cdot 22{,}2 + 19{,}5 \cdot 0{,}8 \cdot 9{,}75 = 1457{,}46 cm^3 \end{cases} \rightarrow W_{ypl.} = 3404{,}56 \text{ cm}^3$$

The plastic strength modulus about the elastic neutral axis (el.n.a.) *Gy* is:

$$\text{El.n.a.}\begin{cases} S_{yt} = 50 \cdot 28{,}1 + 30{,}5 \cdot 0{,}8 \cdot 11{,}96 = 1702{,}32 cm^3 \\ S_{yc} = 58{,}8 \cdot 25{,}49 + 19{,}5 \cdot 0{,}8 \cdot 13{,}04 = 1702{,}24 cm^3 \end{cases} \rightarrow W_{ypl.} = 3404{,}56 \text{ cm}^3$$

The limit plastic moment $M_{lim.pl.}$ is:
$M_{lim.pl.} = W_{y\,pl.} \cdot R_c = 3404{,}56 \cdot 2400 \cdot 10^{-4} = 817$ kNm

The index of efficiency: $i_e = \frac{W_{y.pl.}}{W_{y.el.}} = 1.2$