Chapter 13

TORSION OF THIN-WALLED BARS WHICH HAVE THE CROSS SECTIONS PREVENTED FROM WARPING
(Prevented or non-uniform torsion)

13.1 GENERALS

In our previous chapter named Pure (uniform) Torsion, it was assumed that when a torque M_{t0} twists the member, all cross sections were completely free to warp. But we remember that a rectangular cross section presents, besides the distortion of the cross section, a distortion of the median line. (Fig. 13.1)

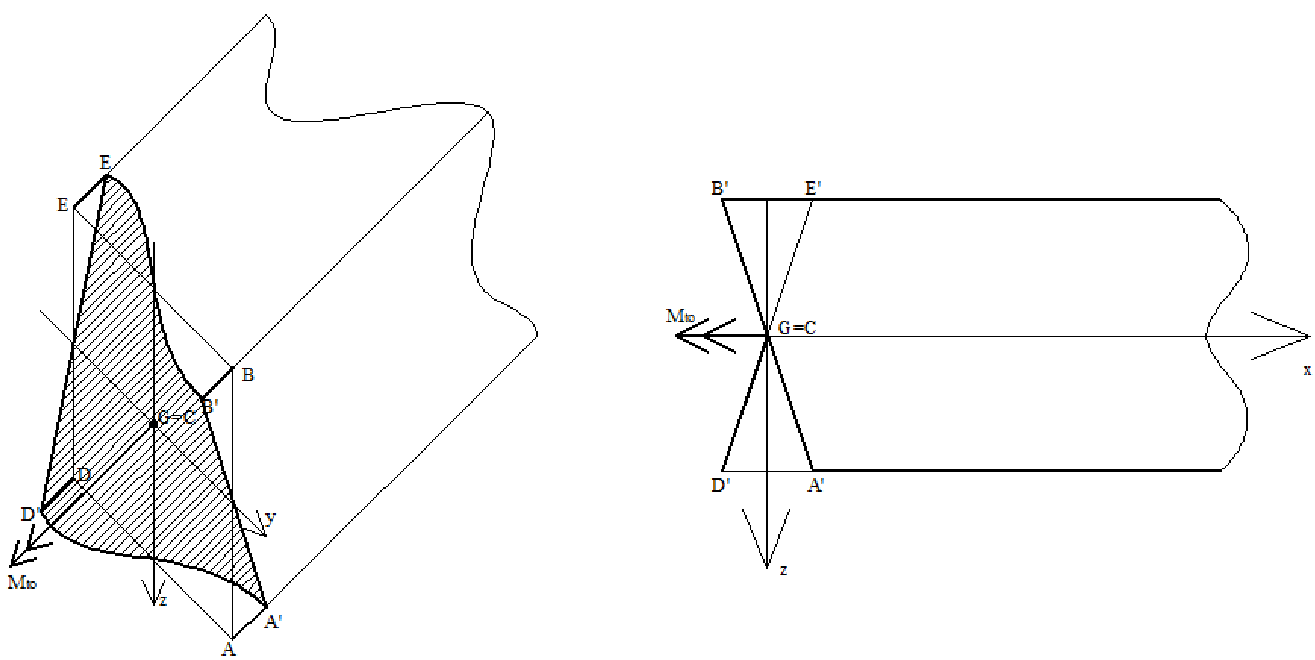


Fig.13.1

From torsion the initial rectangular cross section become a hyperbolic paraboloid (saddle surface) inscribed into a skew rectangle. The median line (*Gz* axis) of the cross section remains in the initial plane of the cross section, so in the median plane of the section the specific sliding is null: $(\gamma_{xy})_{x=0} = 0$

Let's see now a bar with an I-section subjected to torsion by two equal torques M_{t0} acting in each end of the bar (Fig. 13.2).

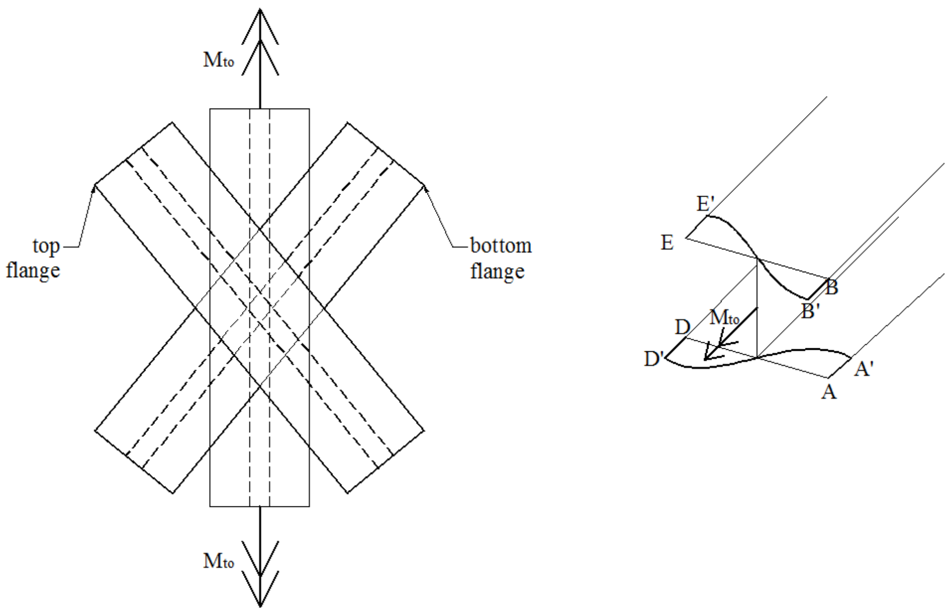


Fig.13.2

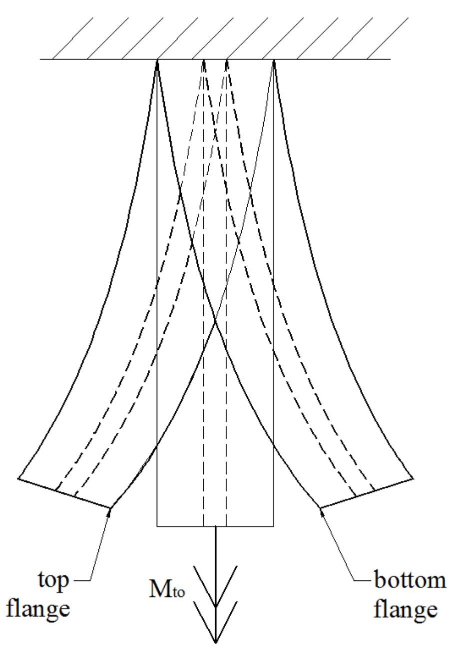


Fig.13.3

Once again, besides the cross section distortion (of the flanges and web) a distortion of the median line appears, but identically for all cross sections. That's why we call this **torsion: free or pure or uniform.**

Now we consider the same bar from Fig.13.2, but with one end fixed (Fig.13.3). We observe that the fixed support introduces a new deformation, the distortion of the cross section. Due to the fixed support a new deformation will appear, the distortion of the median line varying along the bar from zero in the fixed support to a maximum value in the free end. In this case we discuss about **prevented or non-uniform torsion.**

The preventing of the median line distortion is in fact the preventing of the elastic deformations of the points from the median line along the bar axis *Gx*. For this reason new **normal stresses** σ_ω will appear. Because the preventing of this distortion varies along the bar axis, these stresses σ_ω also vary and for this reason they must be equilibrated by new **tangential (shear) stresses** τ_ω**.**

From this prevented deformation, in cross section will appear, only from torsion, 3 new distinct unit stresses:

1. ***Warping normal stresses*** σ_ω produced by the flanges bending (Fig 13.4a). This bending of the flanges doesn't introduce any known stress, because their distribution is equilibrated in section. Anyway, V.Z.Vlasov introduced a new resultant stress, fictitious, called ***bi-moment***, noted with ***B***. For the I-section, this bi-moment may be written as the moment M from each of the flanges multiplied by the distance between them:

$B = M\,(h\text{-}t)$ It has an unusual unit [daN×cm^2], [kN×m^2]

For other type of cross sections the *bi-moment B* isn't so simple to be interpretated. Generally the bi-moment is the distribution of axial stresses that is needed to reduce the warping of the section.

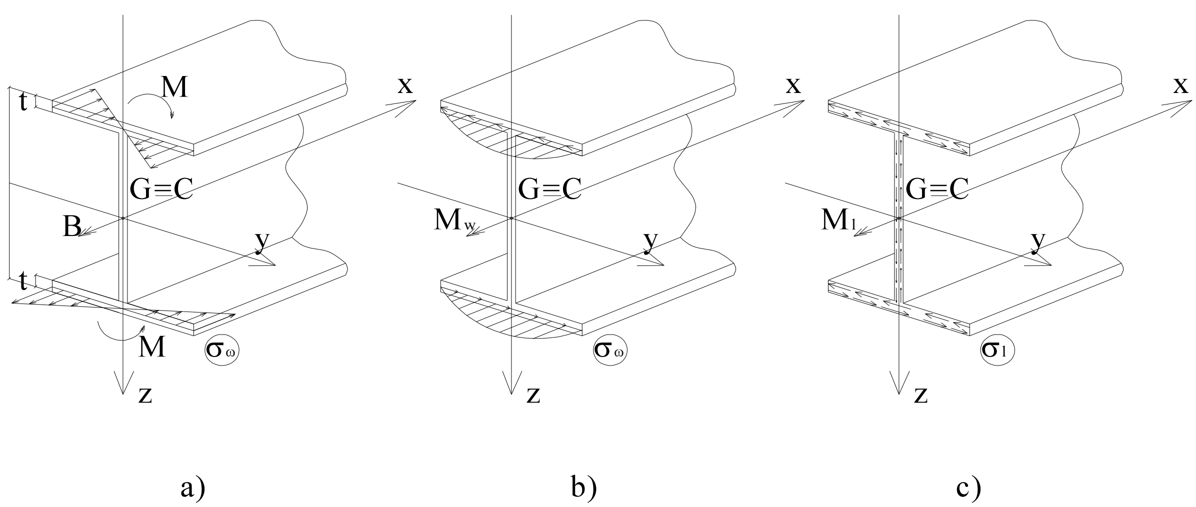


Fig.13.4

2. ***<u>Warping shear stresses</u>*** $\boldsymbol{\tau_\omega}$ are produced by the distortion of the median line of the cross section along the bar (Fig 13.4.b). The resultants of these stresses τ_ω in both flanges, reduced in the shear center C define a moment of torsion $\mathbf{M_\omega = M_{ts}}$, ***called moment of warping (prevented or non-uniform) torsion, or secondary torsional moment***

3. Besides this moment $\mathbf{M_\omega}$, exist also a ***moment of uniform (free) torsion*** $\mathbf{M_{tp}}$, ***or primary torsional moment***, which produce free (uniform) torsion. Both moments M_{ts} and M_{tp} form the total moment of torsion M_t that subject the member:

$$M_t = M_{ts} + M_{tp}$$

M_{tp} will produce torsional shear stresses τ_x calculated in accordance to the chapter of free torsion (Fig .13.4c)

From these 3 stresses only the last stress τ_x is known $\left(\tau_x = \frac{M_{tp}}{W_t}\right)$. To express the unit stresses σ_ω and τ_ω, the deformations produced by prevented torsion must be studied.

13.2 THE NORMAL STRESS σ_ω

It is accepted that the unit stresses σ_ω and τ_ω are **constantly** distributed on the wall **thickness**, equivalent to the constant distribution on thickness of the specific deformations. That's why the study of the deformations is made considering the member composed from the cross section median line and the length of the longitudinal axis of the bar.

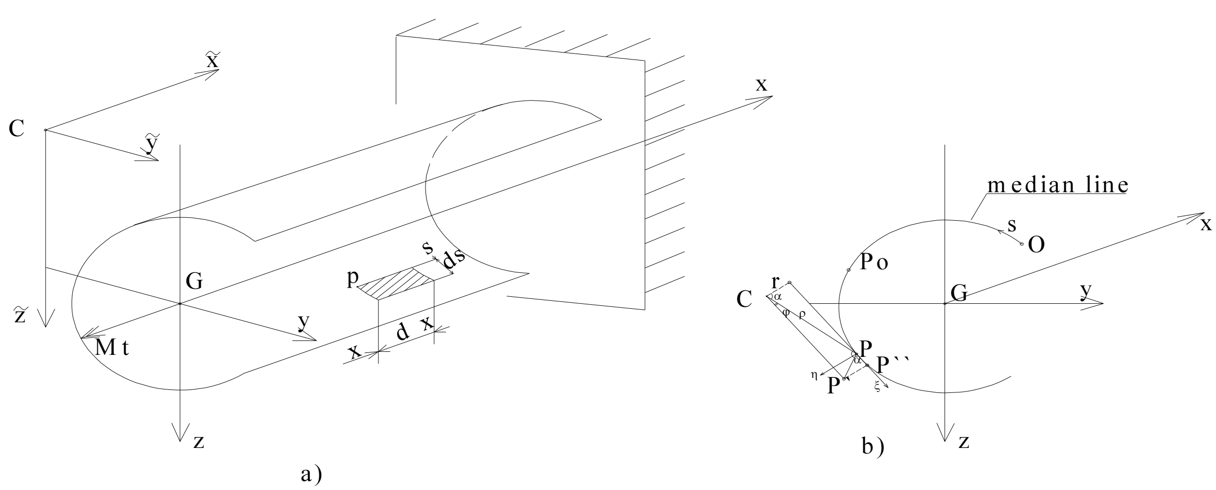


Fig.13.5

Let's consider a thin-walled bar subjected to torsion (Fig 13.5.a) A point P from the median line may be positioned by the coordinates x, y, z from the principal inertia system of axis and by the curvilinear coordinate s, measured along the median line.

From torsion the cross section is twisted around the shear center C with the angle φ (Fig.13.5.b), very small in practice. Therefore the current point P from the median line will have a displacement perpendicular to the radius ρ, PP' (Fig.13.5.b). Point P is characterized also by the system of axis $\xi P \eta$ (ξ is tangent to median line and η is perpendicular to the median line). The component ξ is, from ΔPP'P'':

$\xi = PP'' = PP' \cdot \cos\alpha$ (1)

From ΔCP'P: $\operatorname{tg}\varphi \cong \varphi = \frac{PP\prime}{CP} = \frac{PP\prime}{\rho} \Rightarrow PP' = \rho \cdot \varphi$ (2)

Replacing (2) in (1)

$\xi = \rho \cdot \varphi \cdot \cos\alpha = r \cdot \varphi$ (3)

At the beginning of this chapter: $\gamma_{xz} = 0$. From Cauchy's relation: $\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$

We write this condition in the current point P, in the plan which is tangent to the median line:

$\gamma_{x\xi} = \frac{\partial u}{\partial s} + \frac{\partial \xi}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial s} = -\frac{\partial \xi}{\partial x}$ (4)

The cross section is constant along the bar and r doesn't depend on x, so:

$\frac{\partial \xi}{\partial x} = \frac{\partial (r \times y)}{\partial x} = r \cdot \varphi'$

In (4): $\frac{\partial u}{\partial s} = - r \cdot \varphi'$

From integration: $u = -\int_{s_0}^{s} (r \cdot \varphi')ds = -\varphi' \int_{s_0}^{s} r\,ds$ (5)

The integration is made along the median line, starting from the point P_0, called **main sectorial point,** until the current point P.

From Fig. 13.6 we observe that:

$$\omega = \int_{s_0}^{s} r\,ds \quad (6)$$

ω represents twice the area described by the origin radius CP_0, the current radius CP' and the median line P_0P(the hatched surface). *ω* is called **sectorial coordinate** (or sectorial surface).

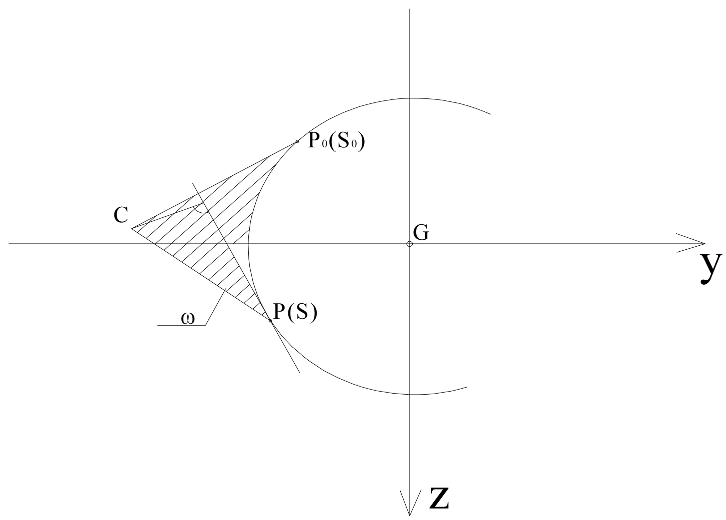


Fig. 13.6

The **sectorial coordinate ω** is positive when the origin radius CP_0 rotates clockwise until the current radius *CP*, measured in [L^2]. With (6), the displacement u from (5) is:

$$u = -\varphi' \cdot \omega \quad (7)$$

From Cauchy's relation:

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\varphi'' \cdot \omega \quad (8)$$

From Hook's law:

$$\sigma_x = E \cdot \varepsilon_x = \sigma_\omega \Rightarrow \boldsymbol{\sigma_\omega = -E \cdot \varphi'' \cdot \omega} \quad (9)$$

Relation (9) is still unknown because the function of the twisting angle φ is unknown, as well as the position of points *C* and P_0 that define the sectorial coordinate *ω*.

From the **static aspect** written for the cross section from figure 13.5 the single stress different from 0 is the torque M_t. Making the other stresses, which produce normal stress σ, equal to zero (from static and strength calculus) we obtain:

- From static calculus:
  $N = M_y = M_z = 0$ (10)
- From strength calculus:
  a) $N = \int_A \sigma_x \cdot dA = \int_A \sigma_\omega \cdot dA = - E \cdot \varphi'' \int_A \omega \cdot dA = -E \cdot \varphi'' \cdot S_\omega = 0$ (11)
  As $E \cdot \varphi'' \neq 0 \Rightarrow S_\omega = 0$ (12)

S_ω is named **sectorial static moment** (or first sectorial moment of area), expressed in $[L^4]$

  b) $M_y = \int_A \sigma_x \cdot z \cdot dA = \int_A \sigma_\omega \cdot z \cdot dA = - E\varphi'' \int_A \omega \cdot z \cdot dA = -E \cdot \varphi \cdot S_{\omega y} = 0$ (13)
  $S_{\omega y} = \int_A \omega \cdot z \cdot dA = 0$ (14)
  $S_{\omega y}$ is called **linear sectorial static moment**, expressed in $[L^5]$
  c) $M_z = \int_A \sigma_x \cdot y \cdot dA = -E \cdot \varphi'' \cdot S_{\omega z} = 0$ (15)
  $S_{\omega z} = \int_A \omega \cdot y \cdot dA = 0$ (16)

Equations (12),(14),(16) permit the complete definitions of sectorial coordinate ω. Equation (14) and (16) define the coordinates of the **shear center *C*.** Equation (12) defines the position of the main sectorial point P_0.

The bi-moment B can't be defined from a static calculus, so Vlasov expressed it from a strength calculus, as:

$$B = \int_A \sigma_\omega \cdot z \cdot dA \qquad (17)$$

Replacing σ_ω from (9):

$$B = - E \cdot \varphi'' \int_A \omega^2 \cdot z \cdot dA \qquad (18)$$

Noting with: $I_w = \int_A \omega^2 \cdot dA$ (19) , relation (18) is written:

$$B = - E \cdot I_\omega \cdot \varphi'' \qquad (20)$$

I_ω is called **sectorial moment of inertia** and it is obtained integrating sectorial coordinate ω with itself. For this reason I_ω is always positive and it is expressed in $[L^6]$.

From (20) we observe that the bi-moment B is function the twisting angle φ. Expressing $E\cdot\varphi''$ from (20) as $E\cdot\varphi'' = -\frac{B}{I_\omega}$, which replaced in (9) gives the final relation of σ_ω:

$$\boldsymbol{\sigma_\omega = \frac{B}{I_\omega}\cdot\omega} \quad (21)$$

13.3 THE SHEAR (TANGENTIAL) STRESS τ_ω

From the thin-walled bar from Figure 13.5a, we isolate a differential element through two cross sections at distance *dx* and at the distance *s* along the median line (Fig.13.7).

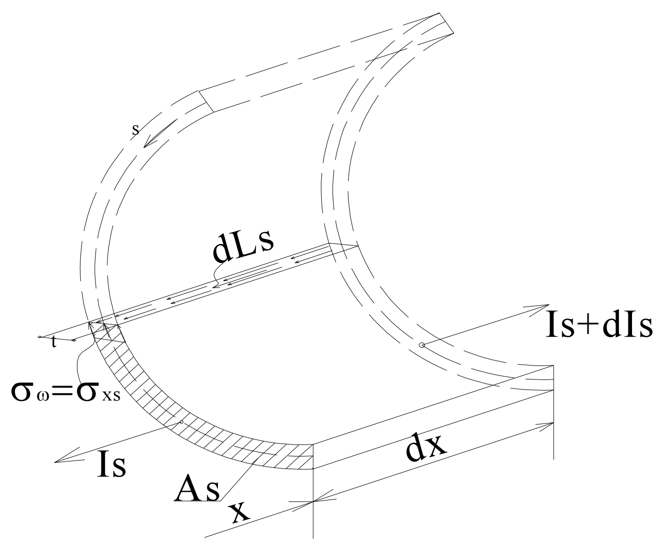


Fig. 13.7

From figure 13.7, the resultant of the normal stress σ_ω from surface A_s, is:

$$I_s = \int_{A_s} \sigma_\omega \cdot dA \qquad (22)$$

At distance *dx*, the resultant of σ_ω is $I_s + dI_s$. On the longitudinal face the resultant of τ_ω (constant on thickness *t*), is : $dL_s = \tau_\omega \cdot t \cdot dx$ (23)

From the equilibrium condition, written about *x* axis:

$$-I_s + I_s + dI_s - dL_s = 0 \Rightarrow dI_s = dL_s \qquad (24)$$

Equation (24) shows that the increasing of the normal stress σ_ω along *x* axis is equilibrated by the appearance of the shear stress τ_ω in longitudinal section. The left term from (24) is expressed from (22) and (21) as:

$$I_s = \int_{As} \sigma_\omega \cdot dA = \frac{B}{I_\omega} \int_{As} \omega \cdot dA = \frac{B \cdot S_\omega}{I_\omega}$$

S_ω is the sectorial static moment of the area A_s. Taking into account that S_ω is constant along the member, dI_s is:

$$dI_s = \frac{S_\omega}{I_\omega} \cdot dB \quad (25)$$

Replacing (25) in (24) and with (23):

$$\frac{S_\omega}{I_\omega} \cdot dB = \tau_\omega \cdot t \cdot dx \quad \Rightarrow \tau_\omega = \frac{\frac{dB}{dx} \cdot S\omega}{t \cdot I\omega}$$

Noting with $M_\omega = \frac{dB}{dx} = M_{ts}$ (26) , ***the moment of prevented torsion or secondary torsional moment***, τ_ω is written finally:

$$\boldsymbol{\tau_\omega = \frac{M\omega \cdot S\omega}{t \cdot I\omega}} \quad (27)$$

13.4. THE DIFFERENTIAL EQUATION OF THE TWISTING ANGLE

Both unit stresses σ_ω (21) and τ_ω (27) are function the bi-moment *B* (20) and respectively the moment of warping torsion M_ω (26) , which depends on the twisting angle φ. If we find the expression of φ we may find *B* and M_ω and then σ_ω and τ_ω .The moment of free torsion or primary torsional moment M_{tp} is written (from previous chapter):

$$M_{tp} = \theta \cdot GI_t = \varphi' \cdot GI_t \qquad (28)$$

The moment of warping torsion M_ω is written from (26) and with (20):

$$M_\omega = M_{ts} = -EI_\omega \cdot \varphi''' \qquad (29)$$

Introducing (28) and (29) in:

$$\boldsymbol{M_t = M_{ts} + M_{tp}} \Rightarrow M_t = -EI_\omega \cdot \varphi''' + GI_t \cdot \varphi' \quad (30)$$

Derivating once again with respect to *x*, in the left side of the equation we find:

$\frac{dM_t}{dx} = -m_t$ (31) , a similar differential relation with the first differential relation between stresses and loads ($\frac{dV}{dx} = -p_n$ or $\frac{dN}{dx} = -p_t$)

Derivating (30) and with (31):

$$-m_t = -EI_\omega \cdot \varphi^{IV} + GI_t \cdot \varphi'' \quad (32)$$

We divide (32) by EI_ω, and we note:

$$k = \sqrt{\frac{GI_t}{EI_\omega}} \quad (33)$$

Equation (32) becomes:

$$\varphi^{IV} - k^2 \cdot \varphi'' = \frac{M_t}{EI_\omega} \quad (34)$$

Equation (34) is the differential equation of the twisting angle at warping (prevented) torsion. The factor *k* is called **the bar characteristic at flexural torsion**, k^2 being the ratio of the torsional rigidity in pure torsion GI_t to prevented torsion EI_ω : $k^2 = \frac{GI_t}{EI_\omega}$

The solution of the differential equation (34) is:

$$\varphi = \varphi_g + \varphi_p \quad (35)$$

The general solution $\boldsymbol{\varphi_g}$ is mathematically calculated:

$$\varphi_g = \varphi_0 + \varphi_0' \cdot x + \frac{\varphi_0''}{k^2}(\text{ch}kx - 1) + \frac{\varphi_0'''}{k^3}(\text{sh}kx - kx)$$

Replacing φ_0'' from (20) and φ_0''' from (30):

$$\varphi_0'' = -\frac{B_0}{EI_\omega} \quad \text{and} \quad \varphi_0''' = \frac{-M_{t0} + GI_t \cdot \varphi_0'}{EI_\omega}$$

$\varphi_g = \varphi_0 + \varphi_0' \cdot x - \frac{B_0}{k^2 EI_\omega}(\text{chkx} - 1) - \frac{M_{to}}{k^3 EI_\omega}(\text{shkx} - kx) + \frac{k^2 \cdot \varphi_0'}{k^3}(\text{shkx} - kx)$

The general solution φ_g has the final form:

$$\varphi_g = \varphi_0 + \varphi_0' \frac{\text{shkx}}{k} + \frac{B_0}{GI_t}(1 - \text{chkx}) + \frac{M_{to}}{kGI_t}(kx - \text{shkx}) \quad (36)$$

Replacing in (35):

$$\varphi = \varphi_0 + \varphi_0' \frac{\text{shkx}}{k} + \frac{B_0}{GI_t}(1 - \text{chkx}) + \frac{M_{to}}{kGI_t}(kx - \text{shkx}) + \varphi_p \quad (37)$$

In relation (37) φ_0 , φ_0', B_0 and M_{t0} are named parameters in origin (for x=0).

The **particular solution φ_p** is also mathematically calculated, with Cauchy – Krilov method, the conditions: $\varphi_0 = \varphi'(0) = \varphi''(0) = 0$; $\varphi'''(0)=1$

$\varphi_4(x) = \text{sh } kx - kx \ (k^3)$

With the changing of the variable $x \rightarrow x - \xi$,

$\varphi_4(x) \rightarrow \varphi_4(x - \xi) - k(x - \xi) = \text{sh } k(x - \xi) - (x - \xi)$

The particular solution: $\varphi_p = \int_0^x \varphi_4(x - \xi) \cdot \frac{M_t(\xi)}{EI_\omega} d\xi$

$$\varphi_p = \frac{1}{kGI_t} \int_0^x [\text{shk}(x - \xi) - k(x - \xi)] \cdot M_t(\xi) \, d\xi \quad (38)$$

We observe that the general solution φ_g from (36) is function the type of support, while φ_p from (38) is function the type of loading.

For the particular solution φ_p two main types of loading are presented:

- for the load with a uniformly distributed torque m_t (Fig 13.8a):

For interval $0 \leq x < a$: $\varphi_p = 0$

For interval $a \leq x \leq b$: $\varphi_p = \frac{m_t}{k^2 GI_t}\left[\text{ch } k(x-a) - 1 - \frac{k^2 (x-a)^2}{2}\right]$

For interval $b \leq x \leq l$: $\varphi_p = \frac{m_t}{k^2 GI_t}\left[\text{ch } k(x-a) - 1 - \frac{k^2 (x-a)^2}{2} - \text{ch } k(x-b) - 1 - \frac{k^2 (x-b)^2}{2}\right]$

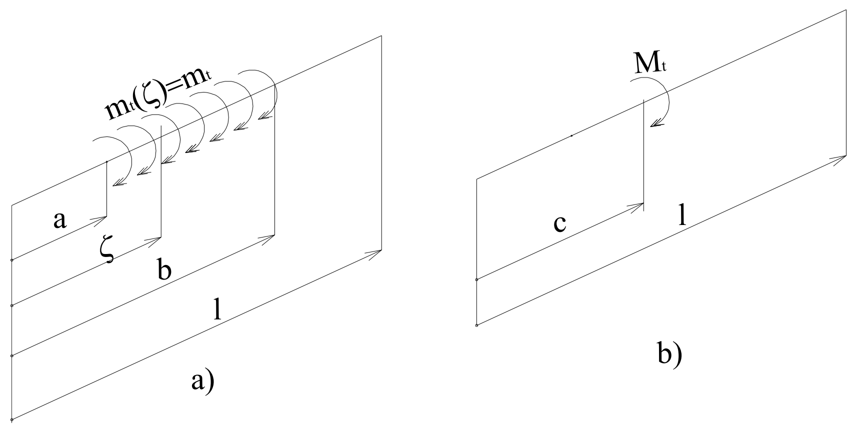


Fig. 13.8

- for the load with a concentrate torque M_t (Fig. 13.8b):

For interval $0 \le x < c$: $\varphi_p=0$

For interval $c \le x \le l$: $\varphi_p=\frac{Mt}{k^2 GI_t}[\text{sh } k(x-c) - k(x-c)]$

To determine the parameters in origin from the general solution φ_g, boundary conditions are written:

- for simple support or hinge:
  $\varphi_0 = 0$ and $B_0 = 0$ (the support prevent the rotation φ but permit the distortion of the median line; from this free distortion $\rightarrow \sigma_\omega = 0 \rightarrow$ from (21) $\rightarrow B = 0$)
- for fixed (built-in) support:
  $\varphi_0=0$ and $\varphi_0'=0$
- for a free end:
  $B_0 = M_{t0} = 0$ (or $B_0=B$; $M_{t0}=M_t$)

13.5 THE SECTORIAL GEOMETRICAL CHARACTERISTICS

13.5.1 The shear center C

To determine the position of C we must choose first an arbitrary center C_1 and an arbitrary main sectorial point P_0'(Fig.13.9).

The center C_1 has the coordinates $C_1(y_{C1}, z_{C1})$

ρ represents the radius C_1P

r is the perpendicular from C_1 to the tangent in P to the median line

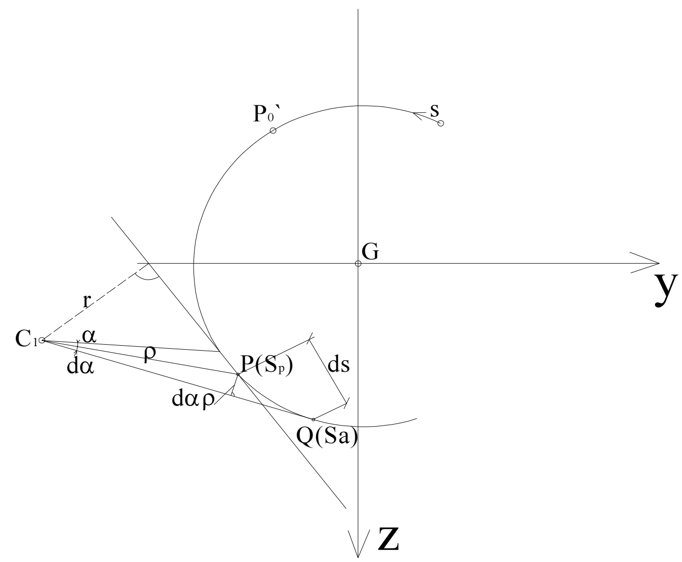
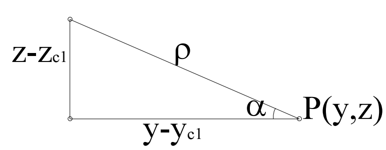


Fig.13.9

From (6) : $\omega_1 = \int_{S_P}^{S_Q} r \cdot ds$

$$d\omega_1 = \rho^2\, d\alpha \qquad (39)$$



$$\rho^2 = (y - y_{C1})^2 + (z - z_{C1})^2 \qquad (40)$$

$y - y_{C1} = \rho \cos\alpha$ and $z - z_{C1} = \rho \sin\alpha$ (41)

The differential of (41):

$dy = -\rho \sin\alpha\, d\alpha = -(z - z_{C1})\, d\alpha$

$dz = \rho \cos\alpha\, d\alpha = (y - y_{C1})d\alpha$ (42)

With (40) and (42) $d\omega_1$ from (39) is:

$d\omega_1 = [(y - y_{C1})(y - y_{C1}) + (z - z_{C1})(z - z_{C1})]d\alpha$

$d\omega_1 = (y - y_{C1})dz - (z - z_{C1})dy$ (43)

For the shear center C, relation (43) is written:

$d\omega = (y - y_C)dz - (z - z_C)dy$ (44)

Extracting (43) from (44):

$d\omega - d\omega_1 = (y_{C1} - y_C)dz - (z_{C1} - z_C)dy$

Integrating:

$\omega = \omega_1 - (y_C - y_{C1})z + (z_C - z_{C1})y + (\text{const.})D$ (45)

We use now the conditions from (14) and (16):

$S_{\omega y} = \int_A \omega\, z\, dA = 0$ (14)

$S_{\omega z} = \int_A \omega\, y\, dA = 0$ (16)

From (16):

$S_{\omega z} = \int_A \omega_1\, y\, dA - (y_C - y_{C1}) \int_A yz\, dA + (z_C - z_{C1}) \int_A y^2\, dA + D\int_A y\, dA = 0$

But : $\int_A y^2\, dA = I_z$; $\int_A yz\, dA = I_{yz} = 0$; $\int_A y\, dA = S_z = 0$

$S_{\omega z} = \int_A \omega_1\, y\, dA + (z_C - z_{C1})I_z = 0$ (46)

In a similar manner, from (14):

$S_{\omega y} = \int_A \omega_1\, z\, dA - (y_C - y_{C1})I_y = 0$ (47)

From (47):

$$y_C = y_{C1} + \frac{\int_A \omega_1 z\, dA}{Iy} \quad (48)$$

From (46):

$$z_C = z_{C1} - \frac{\int_A \omega_1 y\, dA}{Iz} \quad (49)$$

Relations (48) and (49) give the expressions of the coordinates y_C and z_C of the shear center C in a principal system of axis.

13.5.2. The sectorial moment of inertia

$$I_\omega = \int_A \omega^2 \, dA \qquad (50)$$

I_ω is obtained integrating ω with itself. I_ω is always positive and the units are $[L^6]$

13.5.3 The sectorial static moment

$$S_\omega = \int_A \omega \, dA \qquad (51)$$

The integral represents the area of the sectorial coordinate ω and the units are $[L^4]$

13.5.4.The sectorial characteristics for thin-walled members

In relations (48), (49), (50) and (51) the double integrals on surface *A* may be written as linear integrals (as for thin-walled members the thickness *t* is constant), written that:

$$dA = t\, ds \rightarrow \int_A dA = t \int_S ds$$

t: thickness of the thin walls of the member

The relations (48), (49), (50) and (51) become:

$$y_C = y_{C1} + \frac{t \int_S \omega_1 z \, ds}{Iy} \qquad (48')$$

$$z_C = z_{C1} - \frac{t \int_S \omega_1 y \, ds}{Iz} \qquad (49')$$

$$I_\omega = t \int_s \omega^2 \, ds \qquad (50')$$

$$S_\omega = t \int_s \omega \, ds \qquad (51')$$