Chapter 12
PURE TORSION

12.1 GENERALS

A member is subjected to pure torsion if in any cross section of this member the single stress different from zero is the *moment of torsion* or *twisting* (shorter *TORQUE*).
Pure torsion appears when exterior forces acting perpendicular to the bar axis produce only moments of torsion acting along the bar axis (Fig.12.1).

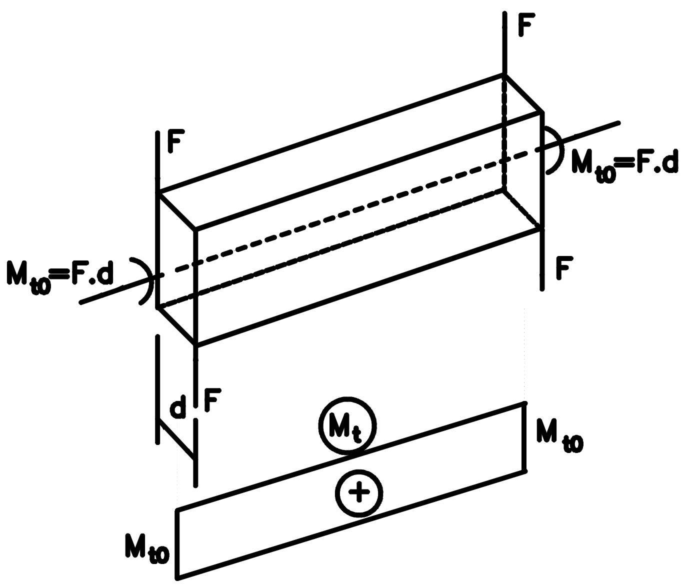


Fig.12.1

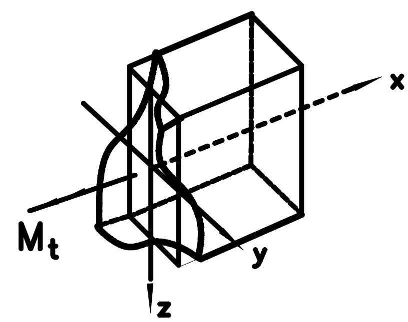


Fig.12.2

The study of torsion is simple for elements with circular or ring-shape cross sections, using completely the hypothesis from Mechanics of Materials. For other types of cross sections: rectangular, sections made from laminated profiles (open or closed), the study is more complex, using the methods from Theory of Elasticity. This is a consequence of the fact that these cross sections are distortion during the member twist. For circular section, due to the symmetry of solicitation, this distortion doesn't appear (the cross sections remain plane during the twist).
For a rectangular section the initial plane section become after twist approximately a hyperbolic paraboloid (Fig.12.2). If the distortion is freely produced the torsion is called *free* or *pure.* If the distortion is prevented from warping we discuss about *prevented* or *ununiform* torsion.

12.2 TORSION OF BARS WITH CIRCULAR SECTION

Let us consider a circular bar, rectangular sheared by parallel circles and equidistant generatrix. Due to the twisting produced by the torques M_t the generatrix are inclined with the same angle, becoming inclined straight lines. The circular sections remain circular during twist and the distances between them do not change. In the rectangles from network only angular deformations γ appear (Fig.12.3).

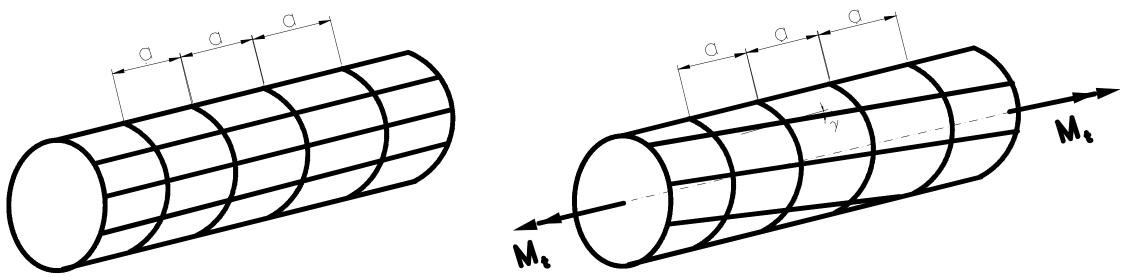


Fig.12.3

The cross sections remain plane after deformation and perpendicular to the bar axis, so the Bernoulli's hypothesis is again certified.

We isolate a differential element from bar (Fig.12.4).

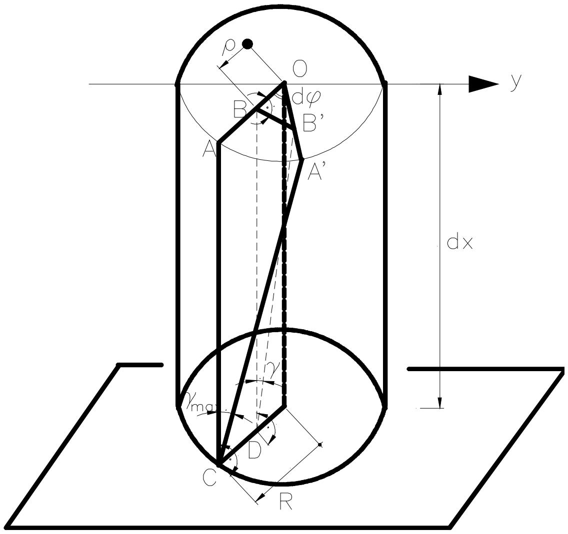


Fig.12.4

Considering the bottom cross section fixed, there will be a rotation of its top cross section through an angle dφ. The angle γ, between the interior generatrix *BD* and the twisted generatrix *B'D*, is the variation of the initial straight angle. As a consequence of the axial symmetry of the deformation, the relative displacement of point *B* is perpendicular to the radius ρ and tangent to the cross section contour.

The specific slipping γ can be expressed from triangles ΔOBB' and respectively ΔDBB', writing the displacement *BB'* of point *B*:

$$BB' = \rho \cdot d\varphi = \gamma \cdot dx$$

and: $\gamma = \rho \cdot \frac{d\varphi}{dx} = \rho \cdot \varphi' = \varphi \cdot \theta$ (1)

with: $\varphi' = \theta = \frac{d\varphi}{dx}$ is the specific twist, representing the angle of twist per unit length of the element.

Since the parallel circles remain at the same distance after twisting, the elongation of the longitudinal fibers is null. So, the specific elongations:

$$\varepsilon_x = 0 \quad (2)$$

Limiting the twisted bar deformations to the elastic domain and using relations (1) and (2) in Hook's laws, we have:

$$\tau_x = G \cdot \gamma = G \cdot \rho \cdot \theta \quad (3)$$

$$\sigma_x = E \cdot \varepsilon_x = 0 \quad (4)$$

The only distinct unit stress is the tangential stress τ, acting perpendicular to the radius R. from (3) we can observe that τ varies directly with the distance ρ, measured from the bar axis; the maximum stresses occur in the outer surface of the bar. The sense of τ is given by sense of the torque M_t (Fig.12.5).

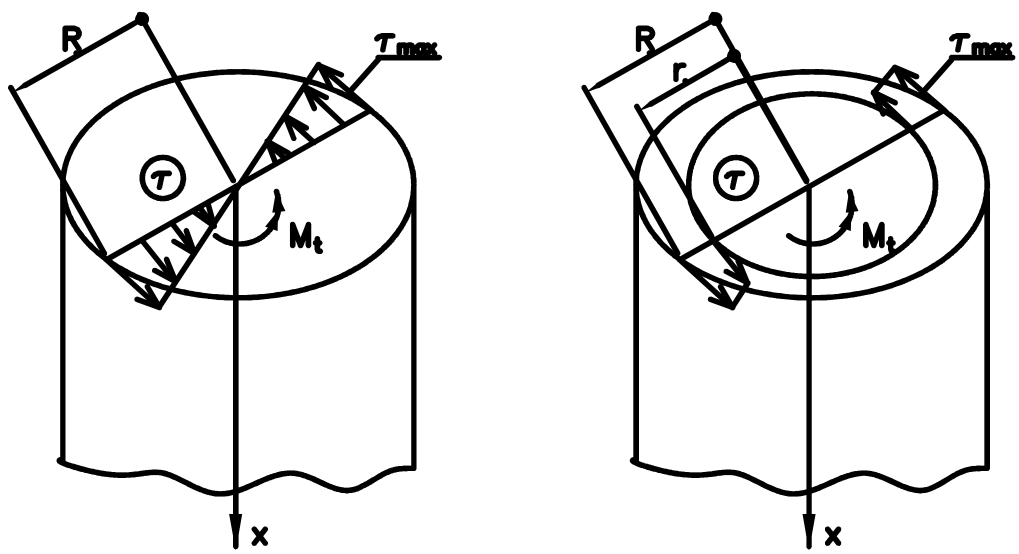
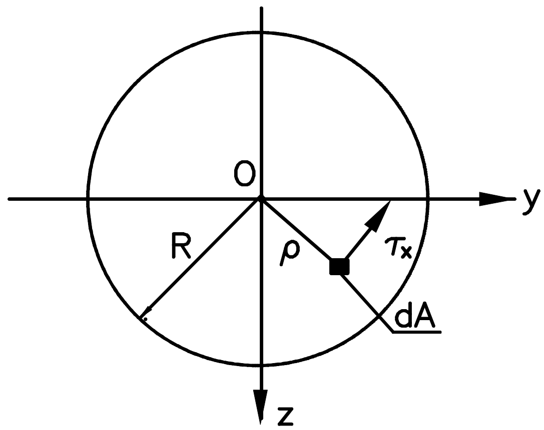


Fig.12.5

From strength calculation the torque M_t is:

$$M_t = \int_A \tau_x \cdot \rho \cdot dA = G\theta \int_A \rho^2 dA = G\theta I_p \quad (5)$$



From (5), the specific twist θ is:

$$\theta = \frac{M_t}{GI_p} \qquad (6)$$

which, replaced in (3), gives the relation of the shear (tangential) stress τ:

$$\tau = \frac{M_t}{I_p}\rho \qquad (7)$$

The term GI_p is the modulus of rigidity in pure torsion (or the torsional rigidity). G is the shear modulus (e.g. for steel $G = 8.1 \times 10^5 daN / cm^2$) and I_p is the polar moment of inertia of a circular cross section (e.g. for circular section with diameter D: $I_p = \frac{\pi D^4}{32}$).

The maximum shear stress corresponds to the maximum radius R:

$$\tau = \frac{M_t}{I_p} R = \frac{M_t}{W_p} \qquad (8)$$

where: W_p is the polar strength modulus for circular sections

The total angle of twist of a bar of length l is:

$$\varphi = \theta \cdot l \Rightarrow \varphi = \frac{M_t \cdot l}{GI_p} \qquad (9)$$

Relation (8) presents similarities with Navier's formula for bent bars, but only from mathematical point of view. It is fundamentally different from this, because it can be applied only to circular cross sections.

12.3. TORSION OF BARS WITH NON-CIRCULAR SECTION

For other type of cross sections the relations obtained in the previous paragraph are no longer valid, because the hypothesis admitted to circular sections can't be used for other type of cross sections. Especially the hypothesis of the plane sections (Bernoulli) isn't valid anymore, because different points of the twisted cross sections have different displacement along the bar axis, the cross sections being distorted (Fig.12.2).

The solution for these twisted bars with non-circular section was given by Barré de Saint-Venant. However, very good results can be obtained making an analogy between the phenomenon of torsion and the phenomenon of the deformation of an elastic membrane. The proceeding is called ***the analogy with the***

elastic membrane, applying the observation that both phenomenons have the same mathematical structure with differential equations with partial derivatives.

Let's assume that in a plate, a hole with the same form and dimensions as the bar cross section is cut. Over this hole a membrane is tensioned by a constant tensile force f [daN/cm^2] on contour (Fig.12.6). This plate with the tensioned membrane becomes the lid of a box, where a gas (e.g.) is introduced under pressure, acting on the box walls with a pressure p [daN/cm^2]. Under these two actions the membrane is deformed becoming a curve surface, the tensile forces from membrane equilibrating the exterior forces p and f.

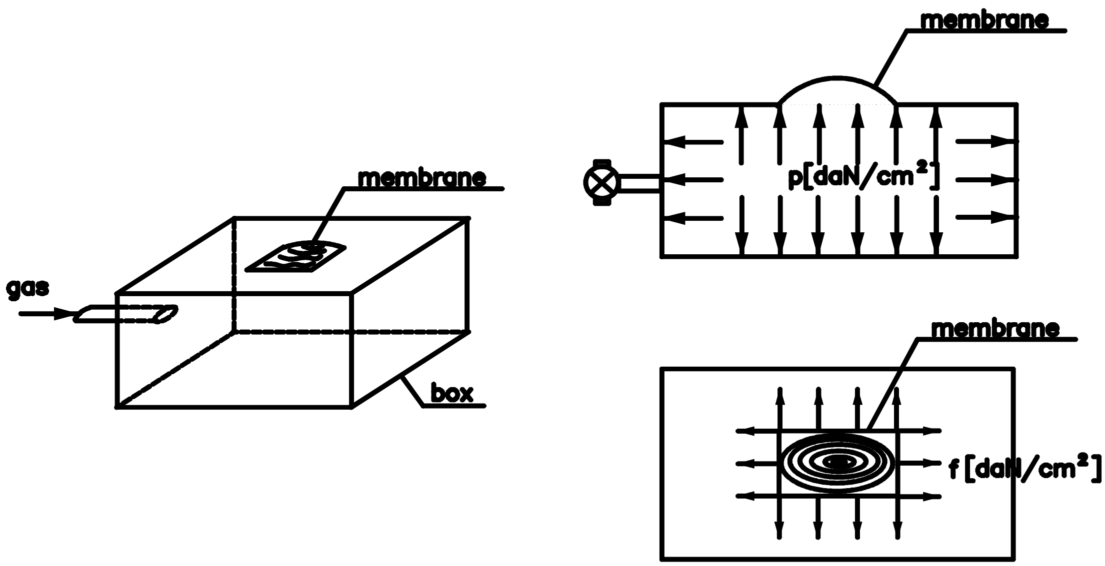


Fig.12.6

It is shown that the differential equation of the deflected surface of the membrane has the same form as the equation which determines the stress distribution over the cross section of the twisted bar. These equations are identically if:

$$\frac{p}{f} = 2G\theta \qquad (10)$$

Three similitudes may be formulated:

a. The tangent to a contour line at any point of the deflected membrane gives the direction of the stress τ_x in the corresponding point of the cross section of the twisted bar.

b. The maximum slope of the membrane in any point is equal to the magnitude of the shear (tangential) stress τ_x in the corresponding point of the twisted bar.

c. The torque M_t of the twisted bar represents twice the volume included between the surface of the deflected membrane and the plane of its outline.

12.3.1 THE NARROW RECTANGULAR CROSS SECTION

A rectangular section is *narrow* if the ratio $h/b \geq 5$ (Fig.12.7). We can neglect the influence of the short sides, so in the membrane analogy the deflected shape of the membrane may be considered as a cylindrical surface having the generatrix parallel to the long sides of the narrow rectangle.

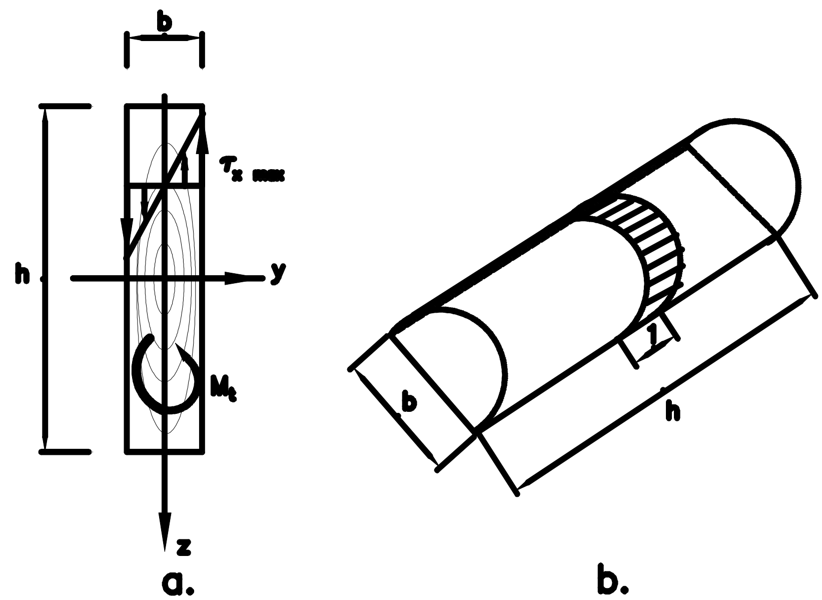


Fig.12.7

We isolate a strip of unit width (Fig.12.7b) and we represent in section the obtained arc (Fig.12.8a).

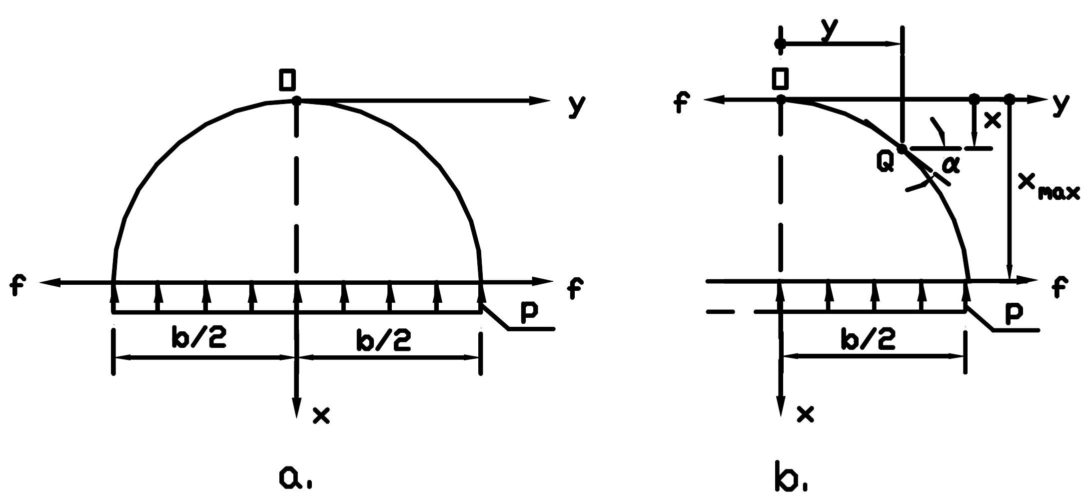


Fig.12.8

The arc has no rigidity in bending, so it can't take over bending moments. We write the expression of the null bending moment in an arbitrary point Q for a half of arc (Fig.12.8b):

$$M_Q = -f \cdot x + p \cdot y \cdot \frac{y}{2} = 0$$

Using condition (10), we find:

$$x = \frac{p}{f} \cdot \frac{y^2}{2} = G\theta \cdot y^2 \qquad (11)$$

Equation (11) is the equation of a parabola of second degree, representing the equation of the surface of the deflected membrane.

The three similitudes may be written:

a. The direction of τ_x is parallel to the long sides of the narrow rectangle (Fig.12.7a)

b. The value of τ_x is the slope of the membrane:

$$\tau_x = tg\alpha = \frac{dx}{dy}$$

or, with (11):

$$\tau_x = 2G\theta \cdot y \qquad (12)$$

The stress τ_x has a linear variation on thickness, with maximum values at $y = \pm\frac{b}{2}$ (Fig.12.7a).

c. The torque M_t is twice the volume:

$$M_t = 2V = 2(A_b \cdot h)$$

with the area of the parabolic surface of the deflected membrane:

$$A_b = 2 \cdot \frac{2}{3} \cdot \frac{b}{2} \cdot x_{\max} = \frac{2}{3} b \cdot x_{\max}$$

and the maximum height x_{max} is for $y = \frac{b}{2}$, with relation (11):

$$x_{\max} = G\theta \left(\frac{b}{2}\right)^2$$

The torque M_t is:

$$M_t = 2 \cdot \frac{2}{3} \cdot b \cdot G\theta \left(\frac{b}{2}\right)^2 \cdot h = G\theta \frac{b^3 h}{3} \qquad (13)$$

and the specific twist is:

$$\theta = \frac{M_t}{GI_t} \qquad (14)$$

where:

$$I_t = \frac{1}{3} b^3 h \qquad (15)$$

I_t is the moment of inertia at free (uniform) torsion for the narrow rectangular sections.
Replacing (14) and (15) in (12) we may calculate the maximum tangential stress $\tau_{x\,max}$, for $y=\frac{b}{2}$:

$$\tau_{x\max}=2G\cdot\frac{M_t}{G\cdot\frac{1}{3}b^3h}\cdot\frac{b}{2}=\frac{M_t}{\frac{1}{3}b^2h}$$

or: $$\tau_{x\max}=\frac{M_t}{W_t} \tag{16}$$

where:

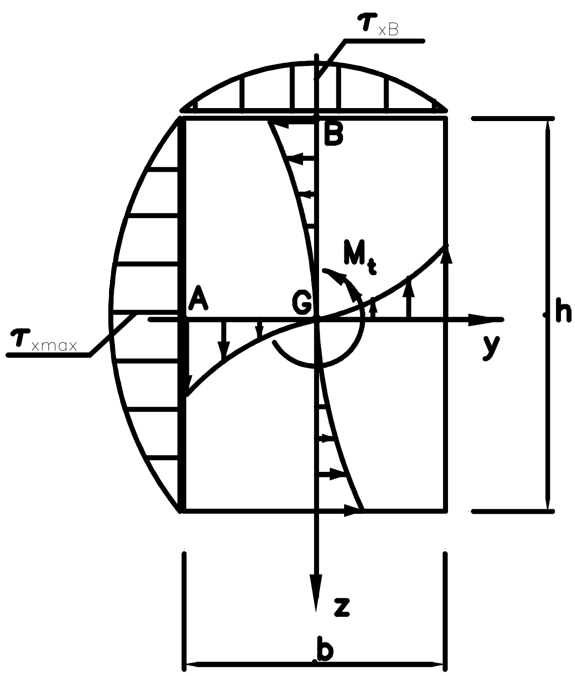
$$W_t=\frac{1}{3}b^2h \tag{17}$$

W_t is the strength modulus at pure torsion for the narrow rectangular cross sections.

12.3.2 THE BROAD RECTANGULAR CROSS SECTION

A broad rectangular cross section is the one that has the ratio $h/b<10$(more severe $h/b<5$) (Fig.12.7).

The tangential stresses distribution is different from the one obtained for the narrow rectangle. The maximum stresses $\tau_{x\,max}$ occur in the middle of the longer sides h (Fig.12.9) and zero in the centroid G and in corners.



Using similar relations, the maximum stress $\tau_{x\,max}$ is, with (16):

$$\tau_{x\max}=\frac{M_t}{W_t} \tag{16}$$

with:

$$W_t=\alpha\cdot b^2h \tag{17}$$

b and h: the shorter, respectively the longer side of the rectangle
α: a numerical factor depending upon the ratio h/b

The tangential stress on the shorter side is:

$$\tau_{x\max}=\frac{M_t}{W_t}\cdot\gamma=\gamma\cdot\tau_{x\max} \tag{19}$$

and the angle of twist per unit length θ is:

Fig. 12.9

$$\theta = \frac{M_t}{GI_t} \quad (14) \qquad \text{with:} \qquad I_t = \beta \cdot b^3 h \quad (20)$$

α, β and γ: are factors given in the table below, for several values of the ratio h/b.

h/b	1,0	1,5	1,75	2,0	2,5	3,0	4,0	5,0	6,0	8,0	10,0	∞
α	0,208	0,231	0,239	0,246	0,258	0,267	0,282	0,292	0,299	0,307	0,313	0,333
β	0,141	0,196	0,214	0,229	0,249	0,263	0,281	0,292	0,299	0,307	0,313	0,333
γ	1,000	0,859	0,820	0,795	0,766	0,753	0,745	0,744	0,743	0,742	0,742	0,742

12.3.3 SECTIONS COMPOSED FROM NARROW RECTANGLES (SIMPLE CONNEX SECTIONS)

Equations (15) and (17) used in (14) and (16) derived for a narrow rectangular cross section, may be used also in other cases in which the width of the cross section is small (cross section made from rolled profiles I, U, L, or composed section which have the median line *open*).

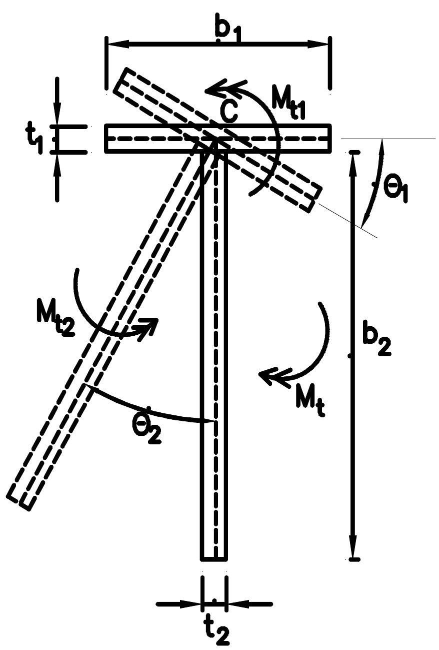


Fig. 12.10

Let's assume we have a *T* shape cross section (Fig.12.10), each rectangle being twisted with the same angle θ.

Each part of the cross section, which is a narrow rectangle, overtakes a part of the torque M_t : M_{t1} and M_{t2}.

From equilibrium condition:

$$M_t = M_{t1} + M_{t2} \qquad (21)$$

From the hypothesis of the cross section undeformability:

$$\theta_1 = \theta_2 \qquad (22)$$

Expressing each angle of twist with (14), relation (22) is written:

$$\frac{M_{t1}}{GI_{t1}} = \frac{M_{t2}}{GI_{t2}} = \frac{M_{t1} + M_{t2}}{G\left(I_{t1} + I_{t2}\right)} = \frac{M_t}{GI_t} \qquad (23)$$

Where with (15):

$$I_t = I_{t1} + I_{t2} = \frac{h_1 \cdot t_1^3}{3} + \frac{h_2 \cdot t_2^3}{3}$$

From (23):

$M_{t1} = M_t \cdot \frac{I_{t1}}{I_t}$ and $M_{t2} = M_t \cdot \frac{I_{t2}}{I_t}$

and the maximum tangential stress in every element

$\tau_{x\ max\ 1} = \frac{M_{t1}}{I_{t1}} \cdot t_1$; $\tau_{x\ max\ 2} = \frac{M_{t2}}{I_{t2}} \cdot t_2$

or replacing M_{t1} and M_{t2} :

$\tau_{x\ max\ 1} = \frac{M_t}{I_t} \cdot t_1$; $\tau_{x\ max\ 2} = \frac{M_t}{I_t} \cdot t_2$

Generally, for a cross section which can be decomposed in *n* narrow rectangles, the maximum shear stress $\tau_{x\ max}$ in the narrow rectangle *i*, is:

$$\tau_{x\ max\ i} = \frac{M_t}{I_t} \cdot t_i \qquad (24)$$

It can be observed that the maximum stress $\tau_{x\ max}$ for the entire cross section will be in the rectangle with the greatest thickness t_{max}:

$\tau_{x\ max} = \frac{M_t}{I_t} \cdot t_{max}$

or, nothing $W_t = \frac{I_t}{t_{max}}$ (25)

which represents the strength modulus at torsion for simple connex sections, the final formula of the maximum shear stress $\tau_{x\ max}$ is:

$\tau_{x\ max} = \frac{M_t}{W_t}$, identically to relation (16)

The angle of twist per unit length θ will be calculated with (14):

$$\theta = \frac{M_t}{GI_t}$$

Where I_t is the moment of inertia at free torsion, for simple connex sections:

$$I_t = \frac{1}{3}\sum_{i=1}^{n} h_i \cdot t_i^3 \qquad (26)$$

n: the number of narrow rectangles which compose the cross section

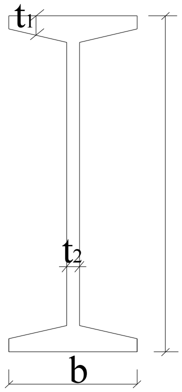
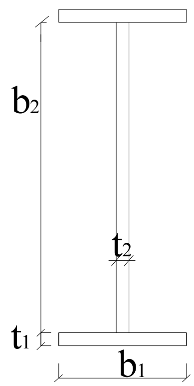
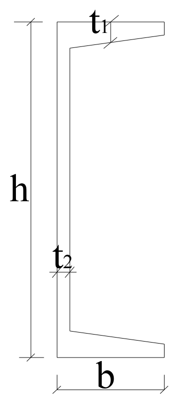
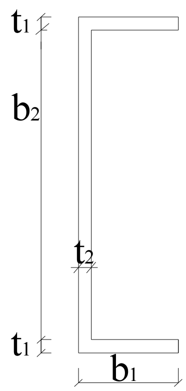
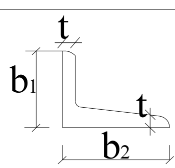
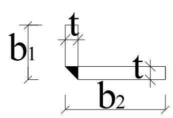
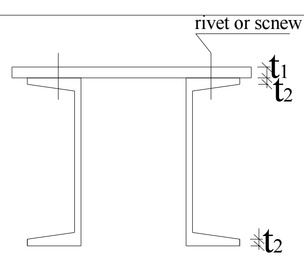
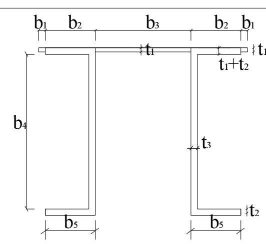
For rolled profiles (IPN, UPN, angle with equal or unequal legs) caused by the reentrant corners (racordarilor colturilor interioare), the formula of I_t is corrected as:

$$I_t = \frac{\eta}{3}\sum_{i=1}^{n} h_i \cdot t_i^3 \qquad (27)$$

where : η is a coefficient calculated from experimental tests.

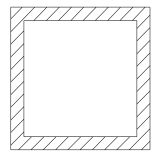
Example for some types of cross sections:

The real cross section / The decomposing into narrow rectangles

	 $\eta=3$; $\eta=1.2$ $I_t=1{,}2/3(2b_1t_1^3+b_2t_2^3)$
	 $\eta=3$; $\eta=1.12$ $I_t=1{,}12/3(2b_1t_1^3+b_2t_2^3)$
	 $\eta=2$; $\eta=1$ $I_t=1/3(b_1t_1^3+b_2t_2^3)$
	 $\eta=9$; $\eta=0.5$ $I_t=0.5/3(2b_1t_1^3+2b_2(t_2+t_1)^3$ $+b_3t_1^3+2b_4t_3^3+2b_5t_2^3)$

For composed welded cross sections: $\eta = 1$

12.3.4 SECTIONS MADE FROM NARROW RECTANGLES, BUT HAVING A CLOSED MEDIAN LINE (DOUBLE CONNEX SECTIONS)



The membrane analogy is again applied. In this case the cross section has two contours and the exterior and interior boundaries of the membrane from Fig.12.11 are located in different horizontal planes.

The device used in membrane analogy (Fig 12.6) is modified as follows: the hole made in the lid of the box will have the same shape and dimension as the cross section ***exterior contour*** (Fig 12.11)**.**

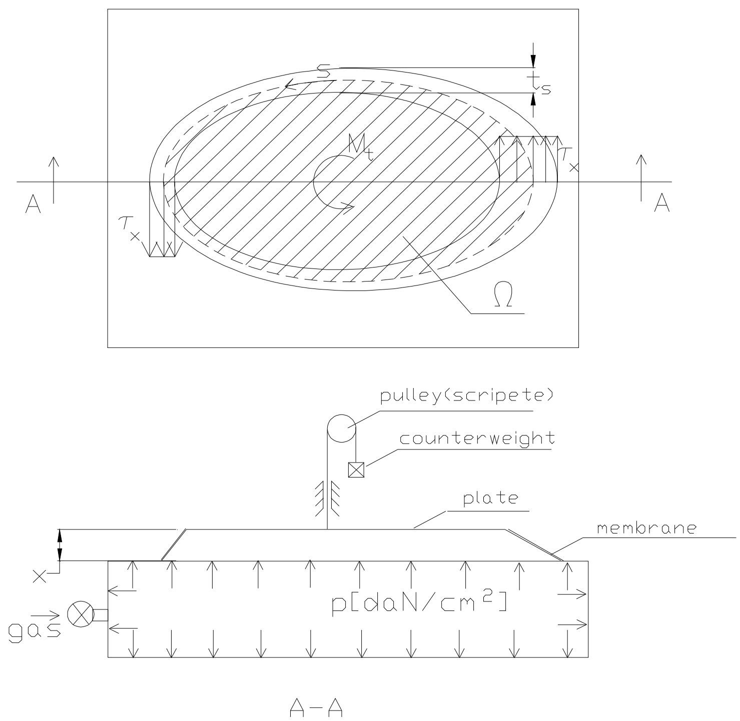


Fig. 12.11

It is made another plate having the shape and dimension of the cross section ***interior contour***. This plate may slip on vertical direction and a balance weight will equilibrate the plate weight. The space between plate and the box lid is covered by the elastic membrane. Under the pressure p[daN/cm^2] the plate riches the new position to a certain height x. The thickness t_s is small and the curvature of the membrane may be neglected, so the deflected surface of the membrane will be a conical surface. The slope of the membrane surface is constant over the thickness t_s and based on the second similitude, the tangential stresses τ_x will be uniformly distributed on the thickness t_s, and they are given by the slope:

$$tg\alpha = \frac{x}{t_s} = \tau_x \qquad (28)$$

The shear stress τ_x direction is tangent to the median line and along the circumference it is inversely proportional to the thickness t_s of the wall.

Using the third similitude, the torque M_t is:

$$M_t = 2V = 2 \cdot \Omega \cdot x \qquad (29)$$

where Ω: is the area bounded by the median line of the cross section (Fig.12.11)

From (29) we may write x:

$$x = \frac{M_t}{2\Omega}$$

which, introduced in (28) gives us the formula of τ_x, called also the relation of Bredt (dated from 1896):

$$\tau_x = \frac{M_t}{2\Omega \cdot t_s} \qquad (30)$$

We observe immediately that the maximum shear stress $\tau_{x\,max}$ correspond to the smallest thickness t_{min}:

$$\tau_{x\,max} = \frac{M_t}{2\Omega \cdot t_{min}}$$

or: $\tau_x = \frac{M_t}{W_t}$, again relation (16)

with: $W_t = 2\Omega \cdot t_{min}$ (31)

W_t: the strength modulus at torsion for double connex sections

To determine the angle of twist per unit length θ, first we have to determine the formula of I_t. That's why we isolate the plate, cutting fictitiously the membrane with a horizontal plane (Fig 12.12)

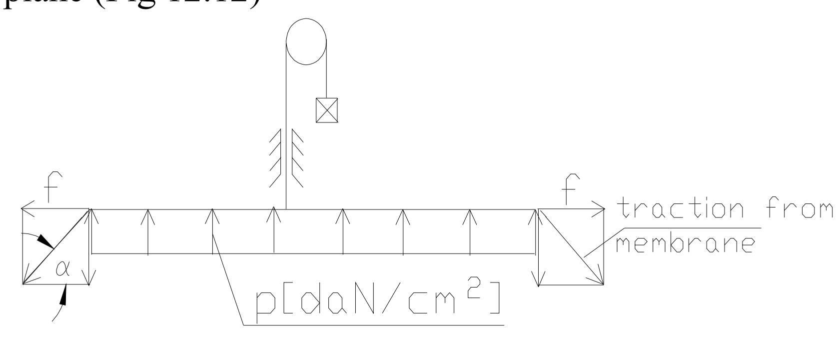


Fig.12.12

We observe that, due to the sliding of the plate only on vertical direction (the plate remains horizontal) and due to the thickness t_s which is variable along the median line s, in membrane are developed tractions f also variable, in what concerns their magnitude and their direction. Writing the equilibrium condition on vertical direction:

$$p \cdot \Omega = \oint (f \cdot ds) \cdot tg\alpha \tag{32}$$

But $tg\alpha = \tau_x$ from (28) => $tg\alpha = \frac{Mt}{2\Omega t_s}$ from (30)

Replaced in (32):

$$p \cdot \Omega = \frac{Mt}{2\Omega} \cdot f \oint \frac{ds}{t_s} \tag{33}$$

Replacing relation (10): $\frac{p}{f} = 2G\theta$ in (33):

$$2G\theta = \frac{Mt}{2\Omega^2} \oint \frac{ds}{t_s}$$

and the angle of twist θ will be :

$$\theta = \frac{Mt}{G \cdot 4\Omega^2} \oint \frac{ds}{t_s}$$

Writing θ with formula (14): $\theta = \frac{M_t}{GI_t}$, we may write the moment of inertia at free torsion (for double connex sections):

$$I_t = \frac{4\Omega^2}{\oint \frac{ds}{t_s}} \tag{34}$$

If the thickness $t_s = t =$ constant => $I_t = \frac{4\Omega^2 \cdot t}{s}$

s: the length of the median line

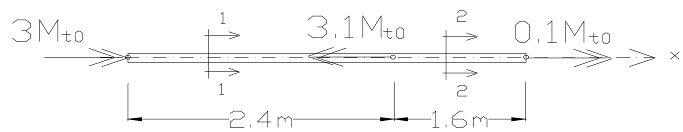
If t_i is constant on s_i => $I_t = \frac{4\Omega^2}{\sum \frac{s_i}{t_i}}$ (35)

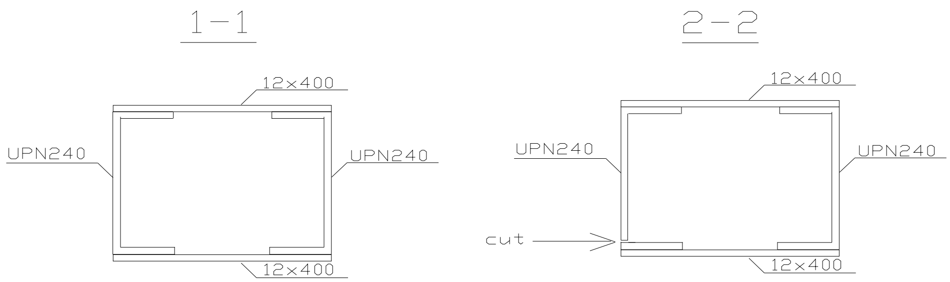
s_i: the length of the median line *i* on which the thickness t_i is constant

12.4 APPLICATION TO PURE TORSION

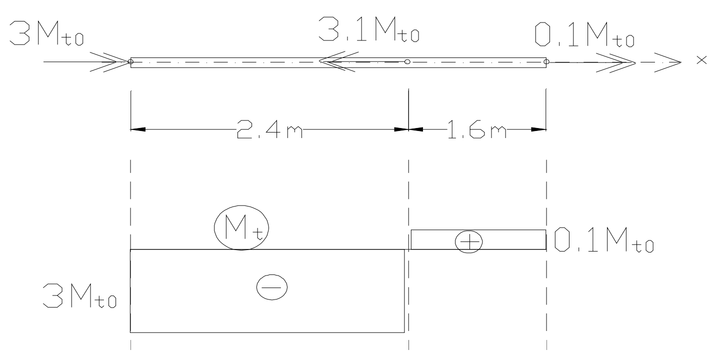
For the following bar having the cross section 1-1 (on the interval of 1.6m length the section is cut, resulting section 2-2) make:

a. M_t diagram
b. Calculate the geometrical characteristics in torsion for both sections
c. Calculate the load parameter M_{t0} from strength and rigidity conditions, if τ_{max}=1300 daN/cm^2 and θ_{max}=0.5 ^{0}/m (G=8.1×10^{5}daN/cm^2)
d. With M_{t0} calculated represent the shear stress τ diagram on each section, inscribing the maximum value
e. With M_{t0} calculated determine the total rotation of the bar (G=8.1×10^{5}daN/cm^2)



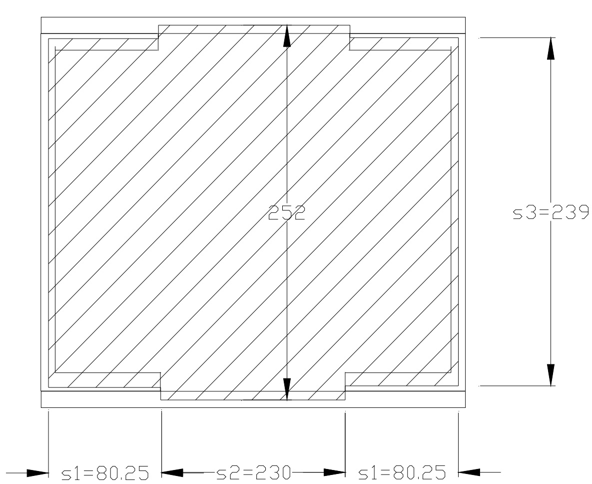


a. M_t diagram:



b. The geometrical characteristics I_t and W_t for each section:

- section 1-1:



[mm]

Surface Ω:

$$\Omega = 25.2\cdot23+2\cdot23.9\cdot8.025 = 963.2\text{cm}^2$$

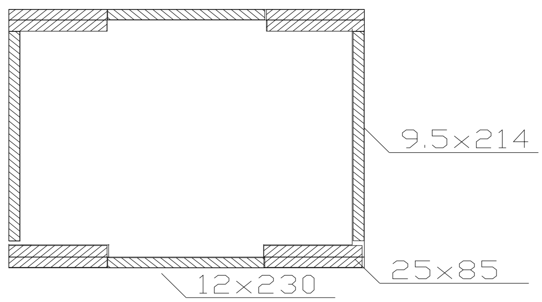
Strength modulus (31):

$$W_t = 2\cdot963.2\cdot0.95 = 1830\text{cm}^3$$

Moment of inertia (35):

$$I_t = \frac{4\cdot963.2^2}{4\cdot\frac{8.025}{2.5}+2\cdot\frac{23}{1.2}+2\cdot\frac{23.9}{0.95}} = 36566\text{cm}^4$$

- section 2-2:



[mm]

Moment of inertia (27) with η=1:

$I_t = \frac{1}{3}(8.5 \cdot 2.5^3 \cdot 4 + 21.4 \cdot 0.95^3 \cdot 2 + $
$+ 23 \cdot 1.2^3 \cdot 2) = 215.8 cm^4$

Strength modulus (25):

$W_t = \frac{215.8}{2.5} = 86.32 cm^3$

c. The load parameter M_{t0} from strength and rigidity conditions:

From strength condition: $M_t \le \tau_{max} \cdot W_t$
From rigidity condition: $M_t \le \theta_{max} \cdot GI_t$

$\theta_{max} = 0.5\ ^0/m = 0.5 \cdot \frac{\pi}{180} \cdot \frac{1}{100} = 8.7266 \times 10^{-5}$ rad/cm

-for section 1-1:

$3M_{t0} \cdot 10^4 \le 1300 \cdot 1830 \rightarrow M_{t0} \le 79.3$ kNm
$3M_{t0} \cdot 10^4 \le 8.7266 \times 10^{-5} \cdot 8.1 \times 10^5 \cdot 36566 \rightarrow M_{t0} \le 86.16$ kNm

-for section 2-2:

$0.1M_{t0} \cdot 10^4 \le 1300 \cdot 86.32 \rightarrow M_{t0} \le 112.22$ kNm
$0.1M_{t0} \cdot 10^4 \le 8.7266 \times 10^{-5} \cdot 8.1 \times 10^5 \cdot 215.8 \rightarrow M_{t0} \le 15.25$ kNm

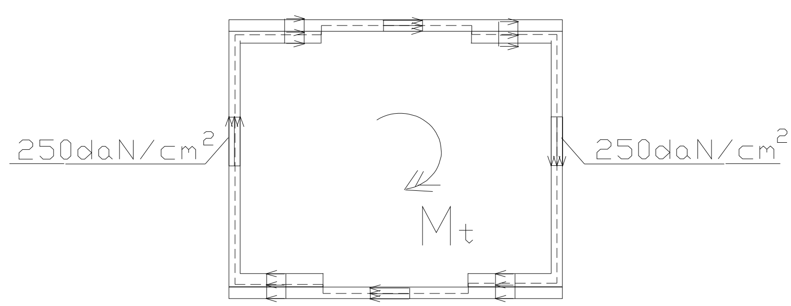
The load parameter M_{t0} is the minimum value from the four values calculated above:

$\underline{M_{t0} = 15.25 \text{ kNm}}$

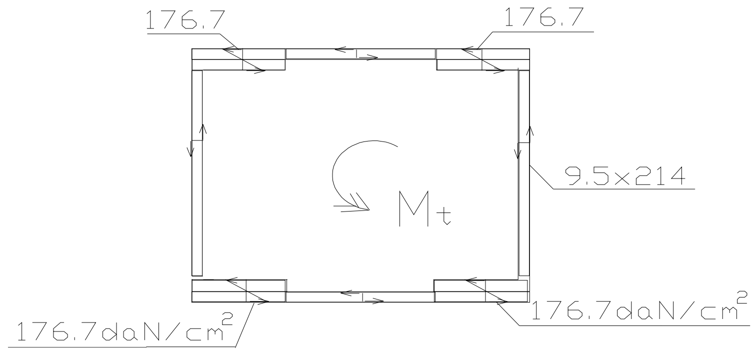
d. The maximum values of the shear stress τ_{max} for each section are:

$\tau_{max\ 1-1} = \frac{3 \cdot 15.25 \cdot 10^4}{1830} = 250$ daN/cm^2
$\tau_{max\ 2-2} = \frac{0.1 \cdot 15.25 \cdot 10^4}{86.32} = 176.7$ daN/cm^2

Shear stress τ diagrams for section 1-1:



Shear stress τ diagram for section 2-2:



e. The total rotation of the bar: $\varphi = \frac{M_t \cdot l}{GI_t}$

$$\varphi_{total} = \frac{-3 \cdot 15.25 \cdot 10^4 \cdot 240}{8.1 \times 10^5 \cdot 36566} + \frac{0.1 \cdot 15.25 \cdot 10^4 \cdot 160}{8.1 \times 10^5 \cdot 215.8} = 0.01025rad = 0.58^0$$