Chapter 11
ENERGETICAL METHOD FOR THE CALCULATION OF DISPLACEMENTS

11.1. INTRODUCTION. HYPOTHESIS

From Theoretical Mechanics we know that, *for a system of material points, the variation of the kinetic energy, written with respect to a fix system of axis, is equal to the sum of the external and internal work*:

$$dE = dL_e + dL_i$$

This theorem of the kinetic energy and work may be written in a finite form:

$$E - E_0 = L_e + L_i \qquad (11.1)$$

where:

E_0: is the initial kinetic energy

E: is the kinetic energy of the system at a given moment

L_e and L_i: is the work produced by external, respectively internal forces, between the initial and final configuration of the system

To study the above theory, the following assumptions are made:

- the material is subjected maximum until the elasticity limit, so it has a perfect elastic behavior; the law of Hooke is valid
- the exterior forces are statically loaded (increasing their intensity slowly, from 0 to the final value)
- the internal frictions and the friction in bearings are neglected
- the work lost by temperature variation is neglected

Considering an *elastic body*, in equilibrium under the action of a system of forces statically applied, the kinetic energy *E* and E_0 will be null, so equation (11.1) is reduced to:

$$L_{total} = L_e + L_i = 0 \qquad (11.2)$$

11.2. THE POTENTIAL ENERGY OF STRAIN

In Theoretical Mechanics the rigid body is considered to be made from material points, the distance between them remaining invariable. Therefore, internal work related to all points of the body is null ($L_i = 0$).

For a deformable body the work is differently evaluated. Let's consider a thin bar, tensioned by exterior forces F applied at both ends (Fig.11.1).

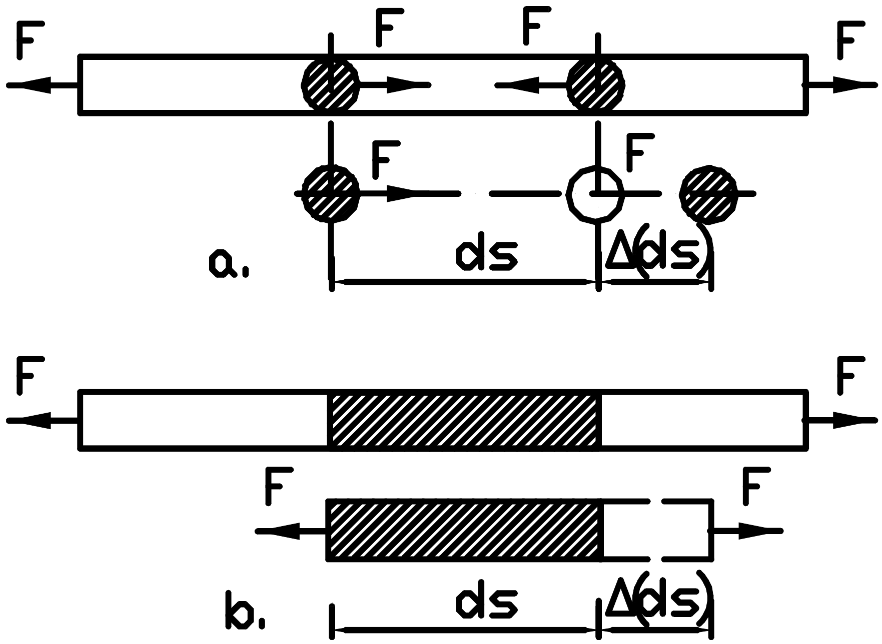


Fig.11.1

Caused by the bar elongation, two material points situated to a distance *ds* one from another, are moving away with a quantity *Δ(ds),* (Fig.11.1a). The attraction forces between these two points will produce the interior work:

$$dL_i = -F \cdot \Delta(ds)$$

with the minus sign, because the force F and the displacement *Δ(ds)* have opposite senses.

Now, consider the same bar as a continuum and by the method of section an element *ds* is isolated. At each ends of this element the axial forces F must be introduced (Fig.11.1b). Being sectional forces, they represent exterior forces for the element *ds*. The work produced by these sectional forces is an exterior work:

$$dL_e = +F \cdot \Delta(ds)$$

being positive because F and *Δ(ds)* have the same sense, but having contrary sign with L_i calculated above:

$dL_i = -dL_e$, or:

$$L_i = -L_e \qquad (11.3)$$

This work, produced by the internal stresses in a deformed body, appears as a consequence of the deformations increasing, and it is called *potential energy of strain*, being noted with U_d.

From (11.3) and (11.2) we have:

$$L_e = U_d \qquad (11.4)$$

Relation (11.4) is called *Clapeyron's first theorem*, being enunciated: *if an elastic body is in rest, the work of the exterior forces is equal to the potential energy* *of strain accumulated by body.* With other words: during the loading of an elementary body the exterior forces will produce a work equal to the variation of the potential energy of these forces.

From (11.4), in what follows the potential energy will be replaced by the exterior work (called also work of strain L_d).

11.3. THE WORK OF STRAIN

From Theoretical Mechanics the work L is defined as the force multiplied by the projection of the displacement along the force direction (Fig.11.2a).

$$L = F \cdot s \tag{11.5}$$

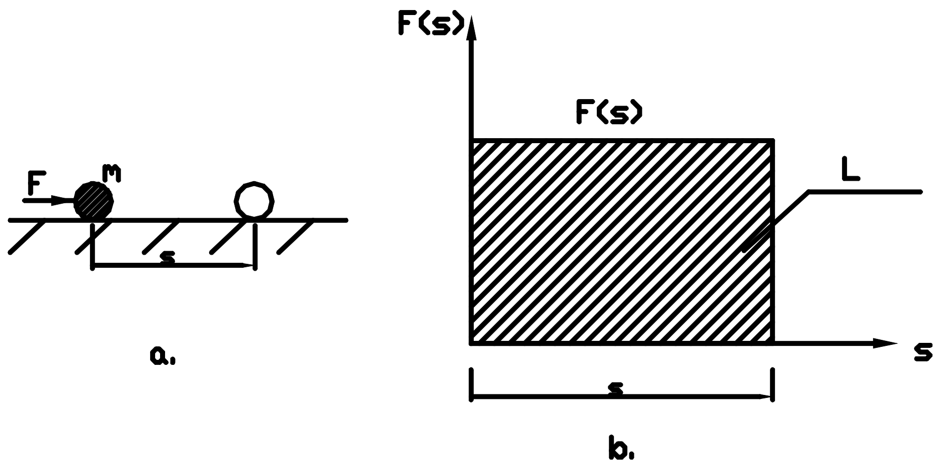


Fig.11.2

Representing the dependence between F and the space covered by it, we obtain a straight line parallel to s axis, the work being the area between this representative curve F(s) and abscissa s (Fig.11.2b).

Let's consider now a deformed bar of length *l* subjected to centric tension (Fig.11.3a).

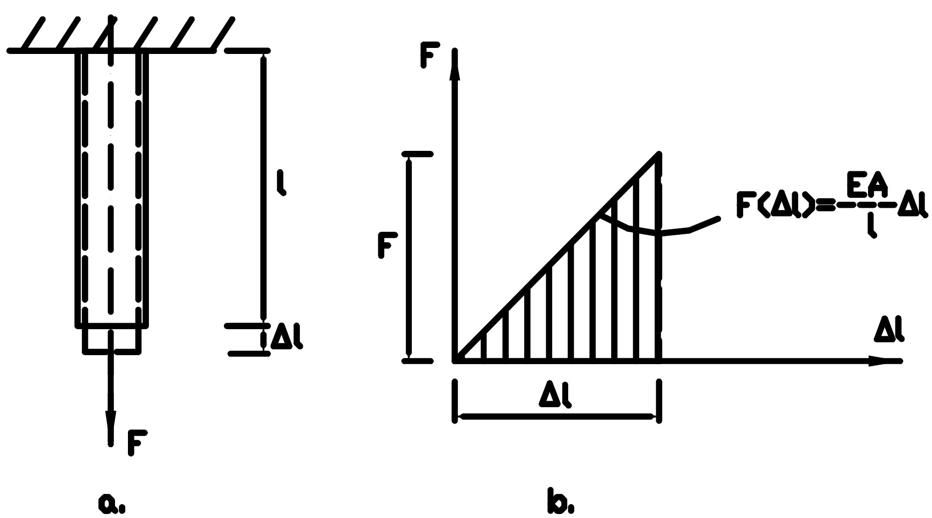


Fig.11.3

The tensile force F produce the displacement (elongation) Δl increasing in value, in the same time with the force F. The work won't be written any more as in equation (11.5): F·Δl, because the values of both factors vary in time. The problem can be written under a differential form. During the bar deformation the force F increase in value with a differential value dF, producing an increasing d Δl of the displacement. The work can be written:

$$dL_e = (F + dF)\cdot d\Delta l \cong F \cdot d\Delta l$$

neglecting the small infinite of second degree $(dF \cdot d\Delta l)$. From integration:

$$L_e = \int F \cdot d\Delta l \qquad (11.6a)$$

But, between F and Δl a proportionality relation (Hook's type) exist:

$$\Delta l = \frac{F \cdot l}{EA} \Rightarrow F = \frac{EA}{l} \cdot \Delta l \qquad (11.6b)$$

In (11.6b) EA/l is a constant for homogeny axial solicitation. Replacing in (11.6a):

$$L_e = \int \frac{EA}{l} \cdot \Delta l \cdot d\Delta l = \frac{EA}{l} \cdot \frac{\Delta l^2}{2} = \frac{EA}{l} \cdot \frac{Fl}{EA} \cdot \frac{\Delta l}{2} = \frac{1}{2} F \cdot \Delta l$$

Resulting from the bar deformation, L_e is called *work of deformation* and we shall note it in what follows with L_d:

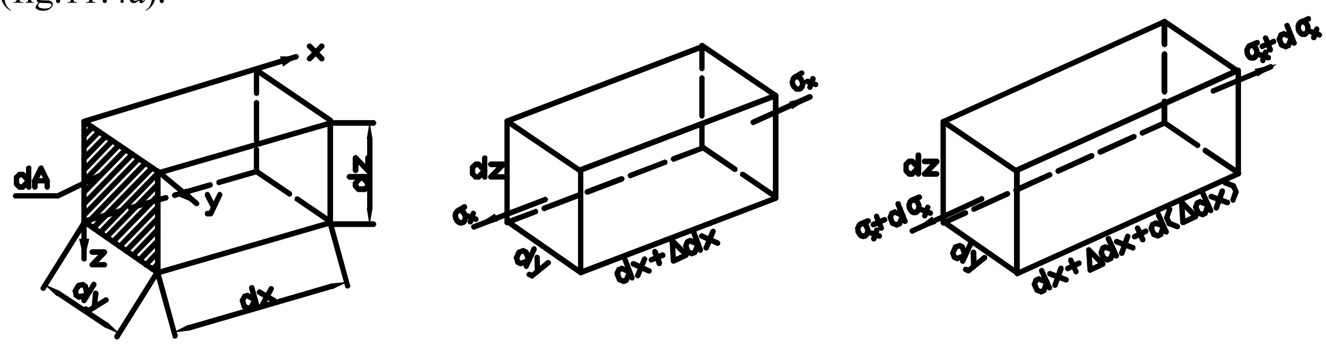
$$\boxed{L_d = \frac{1}{2} F \cdot \Delta l} \text{ (for axial solicitation)} \qquad (11.7)$$

The factor $\frac{1}{2}F$ take account the static application of force, the linear increasing of the force from 0 to the final value (fig.11.3b).

The work of deformation L_d is accumulated in the volume of the bar as potential energy U_d (paragraph 11.4).

11.4. THE POTENTIAL ENERGY OF STRAIN

The strain energy U_d is uniform distributed in the element volume, so it should be calculated first the strain energy accumulated in a differential element of volume (fig.11.4a).



a. b. c.

Fig.11.4

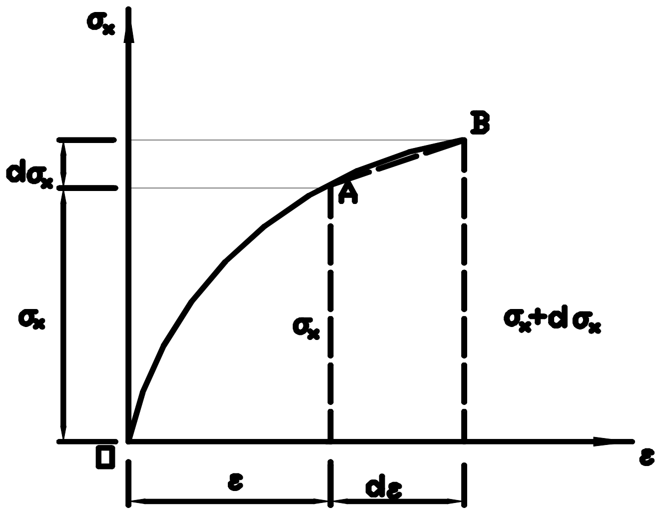


Fig.11.5

Considering an axial solicitation the differential element from figure 11.4a reached the deformed position from figure 11.4b. Assuming that the material has elasto-plastic behaviour, with a non-linear characteristic curve shown in figure 11.5, we have to write the strain energy for a supplementary increasing of load (fig.11.4c), where the linear strain-stresses (σ-ε) variation may be considered.

The arc A-B may be replaced in figure 11.5 by the secant A-B, so the medium values of the unit stresses may be used on this interval.

The second differential of the strain energy $dU_d = \frac{1}{2}(\sigma_x dA)\Delta dx$, will be:

$$d^2U_d = \frac{\sigma_x dA + (\sigma_x + d\sigma_x)dA}{2} \cdot d(\Delta dx) \tag{11.8}$$

Replacing $d(\Delta dx) = d(\varepsilon_x dx) = d\varepsilon_x \cdot dx$ relation (11.8) becomes:

$$d^2U_d = \left(\sigma_x dA + \frac{1}{2} d\sigma_x dA\right) \cdot d\varepsilon_x \cdot dx$$

neglecting the small infinites:

$$d^2U_d = \sigma_x \cdot d\varepsilon_x (dA \cdot dx) = \sigma_x \cdot d\varepsilon_x \cdot dV$$

Integrating twice: once with respect to the specific elongation ε_x and then with respect to the element volume V, we find:

$$U_d = \int_V dV \int_{\varepsilon_x} \sigma_x \cdot d\varepsilon_x \tag{11.9}$$

In (11.9) the second integral represents the elastic strain energy stored into a unit volume (V=1) and it is called the *specific strain energy* (or the *strain energy density*):

$$u_d = \int_{\varepsilon_x} \sigma_x \cdot d\varepsilon_x \tag{11.10}$$

For linear-elastic deformations, Hook's law $\sigma = E \cdot \varepsilon$ is valid, so:

$$u_d = \int_{\varepsilon_x} \sigma_x \cdot d\varepsilon_x = E \int_{\varepsilon_x} \varepsilon_x d\varepsilon_x = \frac{E\varepsilon_x^{\ 2}}{2} = \frac{\sigma_x^{\ 2}}{2E} = \frac{1}{2}\sigma_x \varepsilon_x \tag{11.11}$$

and the strain energy is:

$$U_d = \int_V u_d \cdot dV \tag{11.12}$$

Considering the simultaneous action of the linear specific deformations with respect to x, y and z axis, relation (11.11) become, for linear-elastic deformations:

$$u_d = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z) \tag{11.13}$$

In a similar mode, for angular specific deformations γ_{xy}, γ_{xz} and γ_{yz} relation (11.13) is written:

$$u_d = \frac{1}{2}(\tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz}) \tag{11.14}$$

for linear-elastic deformations, where Hook's law $\tau = G \cdot \gamma$ is also valid.

Finally, the strain energy is written adding (11.13) and (11.14), and with (11.12):

$$U_d = \frac{1}{2}\int_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz}) dV \tag{11.15}$$

or: $$U_d = \frac{1}{2}\int_V [\frac{1}{E}(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2) + \frac{1}{G}(\tau_{xy}{}^2 + \tau_{xz}{}^2 + \tau_{yz}{}^2)] dV \tag{11.16}$$

As the differential volume dV can be written:

$$dV = dA \cdot dx = dx \cdot dy \cdot dz$$

the potential strain energy from relation (11.16) can be written generically:

$$U_d = \frac{1}{2}\iiint (\frac{\sigma^2}{E} + \frac{\tau^2}{G}) dxdydz \tag{11.17}$$

or: $$U_d = \frac{1}{2}\iint (\frac{\sigma^2}{E} + \frac{\tau^2}{G}) dAdx \tag{11.18}$$

11.4.1 The potential strain energy of straight bars, for axial action

For an axially tensioned or compressed bar $\sigma_x = \frac{N}{A}$ and all other unit stresses $\sigma_z = \sigma_y = \ldots = \tau_{yz} = 0$, from (11.18):

$$U_d = \frac{1}{2}\iint \left(\frac{N}{A}\right)^2 \cdot \frac{1}{E} \cdot dAdx = \frac{1}{2}\int_0^l \frac{N^2}{EA^2} dx \int_A dA$$

If the axial force N and the cross sections are constants along the bar, the above integral may be performed, obtaining:

$$\boxed{U_d = \frac{1}{2}\int_0^l \frac{N^2}{EA} dx} = \frac{1}{2} \cdot \frac{N^2}{EA^2} \cdot l \cdot A \tag{11.19a}$$

$$U_d = \frac{N^2 l}{2EA} \tag{11.19b}$$

or $$U_d = \frac{1}{2} N \cdot \Delta l \tag{11.19c}$$

Note that the strain energy U_d from equation (11.19c) is equal to the work of deformation L_d from (11.7), knowing that the axial force N = F constant.

11.4.2 The potential strain energy of straight bars, for bending

For a bent bar, Navier's formula for bending with respect to y axis is: $\sigma_x = \frac{M_y}{I_y} z$, and relation (11.18) become:

$$U_d = \frac{1}{2} \iint \left(\frac{M_y \cdot z}{I_y} \right)^2 \cdot \frac{1}{E} \cdot dAdx = \frac{1}{2} \int_0^l \frac{M_y^{\;2}}{EI_y^{\;2}} dx \int_A z^2 dA$$

as: $\int_A z^2 dA = I_y$, replacing above:

$$\boxed{U_d = \frac{1}{2} \int_0^l \frac{M_y^{\;2}}{EI_y} dx} \qquad (11.20)$$

In the particular case, of constant cross section and bending moment M_y , relation (11.20) is:

$$U_d = \frac{M_y^{\;2} l}{2EI_y} \qquad (11.20a)$$

or, for bending with respect to z axis:

$$U_d = \frac{M_z^{\;2} l}{2EI_z} \qquad (11.20b)$$

11.4.3 The potential strain energy of straight bars, for shearing

Juravski's formula for shearing along z axis is: $\tau = \frac{V_z S_y}{bI_y}$ and the strain energy from (11.18) is:

$$U_d = \frac{1}{2} \iint \left(\frac{V_z \cdot S_y}{b \cdot I_y} \right)^2 \cdot \frac{1}{G} \cdot dAdx = \frac{1}{2} \int_0^l \frac{V_z^{\;2}}{GA} dx \int_A \frac{A \cdot S_y^{\;2}}{b^2 I_y^{\;2}} dA$$

The second integral is a nondimensional coefficient noted with *k*, called *shape factor:*

$$k = \int_A \frac{A \cdot S_y^{\;2}}{b^2 I_y^{\;2}} dA = \frac{A}{I_y^{\;2}} \int_A \left(\frac{S_y}{b} \right)^2 dA \qquad (11.21)$$

This coefficient takes account the nonuniform distribution of shear stresses upon the section height.

With (11.21) the strain energy will be:

$$U_d = \frac{1}{2} \int_0^l k \frac{V_z^{\;2}}{GA} dx \qquad (11.22a)$$

$$\boxed{U_d = \frac{1}{2} \int_0^l \frac{V_z^{\;2}}{G\overline{A}} dx} \qquad (11.22b)$$

where: $\overline{A} = \frac{A}{k}$ is called the *cross section transformed area*

Example: For a cross section (fig.11.6) having a rectangular shape, k is established as follows:

$$I_y = \frac{bh^3}{12} \; ; \; A = bh \; ; \; dA = b \cdot dz$$

$$S_y = b\left(\frac{h}{2} - z\right)\left(z + \frac{h}{4} - \frac{z}{2}\right) = \frac{b}{2}\left(\frac{h^2}{4} - z^2\right)$$

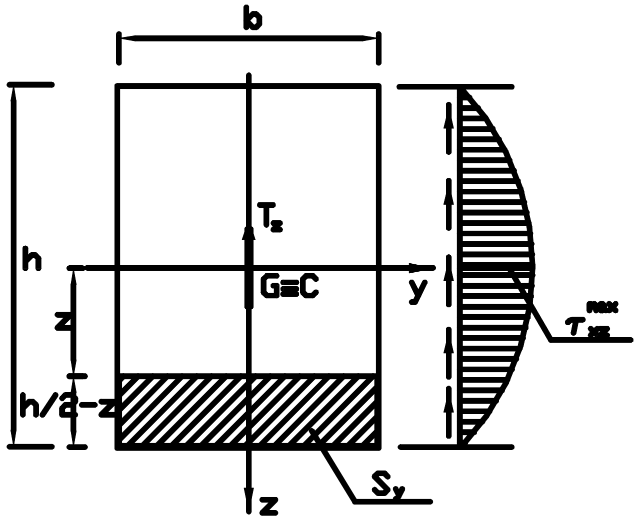


Fig. 12.6

Replacing in (11.21):

$$k = \frac{bh \cdot 12^2}{b^2 h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{4}\left(\frac{h^2}{4} - z^2\right)^2 bdz = 1,2$$

For rolled profiles I or U, the profile web take over the entire shear force. The shape factor can be expressed in this case by the relation:

$k = \dfrac{A}{A_{web}}$, being k = 2,0….2,4

11.4.4 The potential strain energy of straight bars, for torsion

Limiting the explanations to circular or tubular cylindrical bars, having the torsion moment of inertia I_t equal to the polar moment of inertia I_p : $I_t = I_p$ = constant on the entire cross section, the strain energy is, with $\tau = \dfrac{M_t}{I_p} \cdot r$:

$$U_d = \frac{1}{2}\iint\left(\frac{M_t}{I_p} \cdot r\right)^2 \cdot \frac{1}{G} \cdot dAdx = \frac{1}{2}\int_0^l \frac{M_t^2}{GI_p^2} dx \int_A r^2 dA$$

as: $\int_A r^2 dA = I_p$, replacing above:

$$\boxed{U_d = \frac{1}{2}\int_0^l \frac{M_t^2}{GI_p} dx} \qquad (11.23)$$

If the torque M_t and the cross section are constant along the bar axis, equation (11.23) is:

$$U_d = \frac{M_t^2 l}{2GI_p}$$

11.5. ENERGY PRINCIPLES AND THEOREMS

The energetically approach of the strength of materials problems represents a very efficient alternative with respect to the statically approach.
In Theoretical Mechanics which deals only with *nondeformable bodies*, the *principle of virtual work* is used under two distinct forms:

- *The principle of virtual displacement*, also called *principle of virtual work*, when forces applied to the structure are real external forces in equilibrium and the virtual work is produced imposing to the structure virtual displacements
- *The principle of virtual forces*, also called *principle of complementary virtual work*, when the displacements of the structure are real displacements and the virtual work is carried out by the variation of virtual forces applied to the structure.

11.5.1 The principle of virtual displacements

Consider a deformed bar in equilibrium (position (1), in fig.11.7), an infinite close position arbitrary chosen (position (2), in fig.11.7), is imparted, compatible with the geometrical conditions imposed by supports. The new position of the bar is fictitious, virtual. The displacements, between the position in equilibrium and the infinite close position, are called *virtual displacements*.

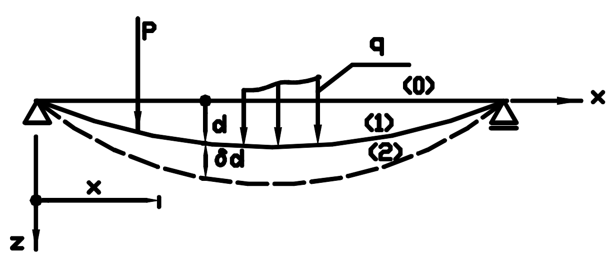


Fig. 11.7

The displacement from the initial position (0) to the equilibrium position (1) is *d* and the virtual displacement, between (1) and (2), is δd.
The system of forces from the static equilibrium position (1) is composed from the exterior forces which deform the bar and the interior forces produced by the bar deformations.

Imparting to the bar a compatible virtual displacement (position (2) from figure 11.7), the principle of virtual work is:

$$\delta L_e + \delta L_i = 0 \qquad (11.24)$$

with: δL_e: the virtual work performed by the exterior forces
δL_i: the virtual work performed by the interior forces

Equation (11.24) represent the energetically condition of equilibrium the system of forces.
Noting with δd_P the virtual displacement with respect to the force P direction, the exterior virtual work is:

$$\delta L_e = \sum P \cdot \delta d_P = \delta_d \sum P \cdot d_P = \delta_d \cdot L \qquad (11.25)$$

with: $L = \sum P \cdot d_P$: the work of final intensity of exterior forces (missing the factor ½ of the static mode of action), the virtual displacement δd_P being a variation of the real displacement d_P .

δ_d: the virtual variation of the deformation elements (specific linear and angular deformations, displacements of the points of application of the exterior forces)

As we explained in paragraphs (11.1) and (11.2), the work of interior forces, from relation (11.3) is:

$$L_i = -L_e$$

and the exterior work L_e is the strain energy U_d , from relation (11.4):

$$L_e = U_d$$

From these two relations, the interior work L_i is:

$$L_i = -U_d \tag{11.26}$$

From (11.12) with (11.13) and (11.14), written under a shorter form:

$$U_d = \int_V u_d \cdot dV = \frac{1}{2}\int_V (\sigma\varepsilon + \tau\gamma)dV = -L_i \tag{11.27}$$

Returning to the initial problem referring to the principle of virtual work, the interior virtual work can be written from (11.27) (neglecting the factor ½ of the static character):

$$\delta L_i = -\int_V (\sigma \cdot \delta\varepsilon + \tau \cdot \delta\gamma)dV = -\delta_d \cdot U_d \tag{11.28}$$

Replacing (11.25) and (11.28) in relation (11.24):

$$\delta_d \cdot L - \delta_d \cdot U_d = 0 \quad \text{or} \quad \delta_d(L - U_d) = 0 \tag{11.29}$$

where: L is the work of final intensity:

$$L = \sum P \cdot d_P \tag{11.30}$$

We note with:

$$\Pi = U_d - L \tag{11.31}$$

Π: represent *the potential or the total potential energy* (the capacity of the deformed body to cede energy)

Relation (11.29) becomes:

$$\delta_d \cdot \Pi = 0 \tag{11.32}$$

Equation (11.32) represents *the principle of a stationary value of the total potential energy*, which can be formulated as: *"Of all compatible displacements satisfying given boundary conditions, those which correspond to the body equilibrium make the total energy Π assumes a stationary value"*. It must be remarked that the linearity of the stress-strain relationship has not been invoked above, so that this principle is valid for nonlinear, as well as linear stress-strain law, as long as the body remain elastic. The external forces must be conservative forces (in which energy is conserved).

11.5.2 The principle of the virtual forces

In the principle of the virtual displacement there were no restrictions concerning the magnitude of the body deformations. In this principle we shall consider that the real deformation of the bar is very small (the hypothesis of the small deformations), this deformation being considered a *virtual displacement.*

Formally in the principle of the virtual displacement we replace δd with d and δd_P with d_P . This deformation is produced by a system of forces (exterior and interior) arbitrary chosen, which fulfill the condition of equilibrium imposed by the principle of virtual work. These forces are generically noted with δP being called *virtual forces.*

The principle of virtual work from (11.24):

$$\delta L_e + \delta L_i = 0$$

is now written:

$$\sum P \cdot d_P - \int_V (\sigma \cdot \varepsilon + \tau \cdot \gamma) dV = 0 \tag{11.33a}$$

where in the expressions of δL_e from (11.25) and δL_i from (11.28) the virtual displacement δd_P, $\delta\varepsilon$ and $\delta\gamma$ were replaced by real values d_P, ε and γ.

Considering now, in the imposed condition of fulfilling the statically equilibrium, that the forces and the unit stresses vary arbitrarily to $P+\delta P$, $\sigma+\delta\sigma$ and $\tau+\delta\tau$, the principle of virtual displacement is:

$$\sum (P + \delta P) \cdot d_P - \int_V [(\sigma + \delta\sigma)\varepsilon + (\tau + \delta\tau)\gamma] dV = 0 \tag{11.33b}$$

Subtracting (11.33a) from (11.33b) we obtain:

$$\sum \delta P \cdot d_P - \int_V (\delta\sigma \cdot \varepsilon + \delta\tau \cdot \gamma) dV = 0 \tag{11.34}$$

Relation (11.34) represent the mathematical expression of the *principle of virtual forces*, which is inverse to the principle of virtual forces, because now the deformation is real and the exterior forces and the unit stresses are virtual.

Similarly to equations (11.29) and (11.32) this principle may be written:

$$\delta_f (U_d - L) = 0 \tag{11.35a}$$

or: $$\delta_f \cdot \Pi = 0 \tag{11.35b}$$

where δ_f : is the virtual variation of the force elements (exterior forces and unit stresses)

11.5.3 Castigliano's theorems

These theorems are results of the *principle of a stationary value of the total potential energy*, being published by Castigliano in 1879.

Castigliano's first theorem is expressed by the relation:

$$\boxed{d_i = \frac{\partial U_d}{\partial P_i}} \qquad (11.36)$$

and it may be stated as follows: *"For a linear structure, the partial derivative of the strain energy with respect to any load P_i is equal to the corresponding displacements d_i, provided that the strain energy is expressed as a function of loads"*

Carlo Alberto Castigliano (1847-1884)

This theorem is very useful for the calculus of displacements in a linear structure, where the stresses or the internal actions may be established statically or with other methods.

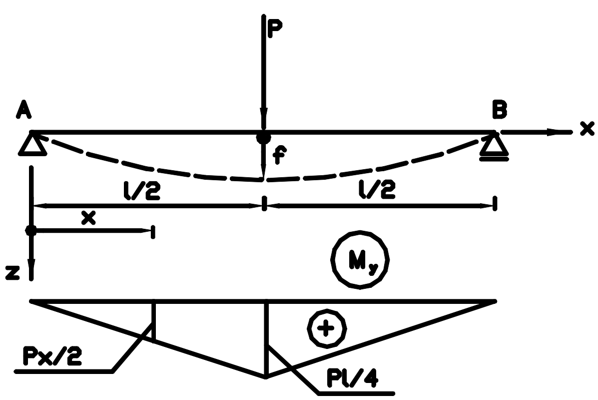


Fig.11.8

Let's apply this theorem, to calculate the maximum deflection *w* of the simple supported beam loaded by a concentrate force P situated at middle span, like in figure 11.8. In the calculus of the strain energy we take account only the effect of the bending moment M_y. From relation (12.20) the strain energy:

$$U_d = \frac{1}{2}\int_0^l \frac{M_y^2}{EI_y}dx$$

The deflection w using relation (11.36) (neglecting the factor ½ of the static character), will be:

$$w = \int_0^l \frac{1}{EI_y} M_y \frac{\partial M_y}{\partial P} dx$$

But as the moment in a section x is $M_y = \frac{P}{2}x$ (fig.11.8), the partial derivative from the above relation is:

$$\frac{\partial M_y}{\partial P} = \frac{\partial}{\partial P}\left(\frac{P}{2}x\right) = \frac{x}{2}$$

Taking account the load symmetry, the deflection w will be:

$$w = \frac{1}{EI_y} 2 \int_0^{l/2} \frac{P}{2} x \cdot \frac{x}{2} dx = \frac{P}{2EI_y} \int_0^{l/2} x^2 dx = \frac{Pl^3}{48EI_y}$$

Castigliano's second theorem is expressed by the relation:

$$\boxed{P_i = \frac{\partial U_d}{\partial d_i}} \qquad (11.37)$$

and it states that: "*The partial derivative of the strain energy with respect to any displacement d_i is equal to the corresponding force P_i, provided that the strain energy is expressed as a function of displacements*"

11.6. MOHR-MAXWELL'S ENERGETICALLY FORMULA FOR DISPLACEMENTS CALCULATION

For linear-elastic members subjected to compound actions, *the strain energy U_d* is the sum of U_d relative to each interior action (equations (11.19a), (11.20), (11.22a), (11.23)):

$$U_d = \frac{1}{2}\sum\left[\int \frac{N^2}{EA}dx + \int \frac{M^2}{EI}dx + \int \frac{V^2}{G\overline{A}}dx + \int \frac{M_t^{\,2}}{GI_p}dx\right] \qquad (11.38)$$

Based on the *Castigliano's first theorem*:

$$d_i = \frac{\partial U_d}{\partial P_i} = \frac{1}{2}\sum\left[\int \frac{N}{EA}\frac{\partial N}{\partial P_i}dx + \int \frac{M}{EI}\frac{\partial M}{\partial P_i}dx + \int \frac{V}{G\overline{A}}\frac{\partial V}{\partial P_i}dx + \int \frac{M_t}{GI_p}\frac{\partial M_t}{\partial P_i}dx\right] \qquad (11.39.a)$$

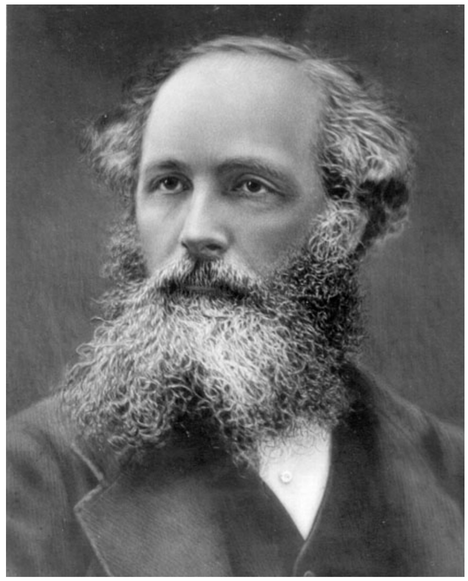
On the other hand, in the case of a linear elastic structure, the internal actions may be expressed as homogenous linear function of the external loads:

$$N = n_1 P_1 + \ldots + n_i P_i + \ldots \quad , \quad V = v_1 P_1 + \ldots + v_i P_i + \ldots \; ,$$
$$M = m_1 P_1 + \ldots + m_i P_i + \ldots \quad , \quad M_t = m_{t1} P_1 + \ldots + m_{ti} P_i + \ldots \qquad (11.39b)$$

where n_i , v_i , m_i , m_{ti} represent the *influence coefficients* and are the internal stresses, in any cross section, produced by a generalized virtual unit force $\left(P_i = \bar{1}\right)$

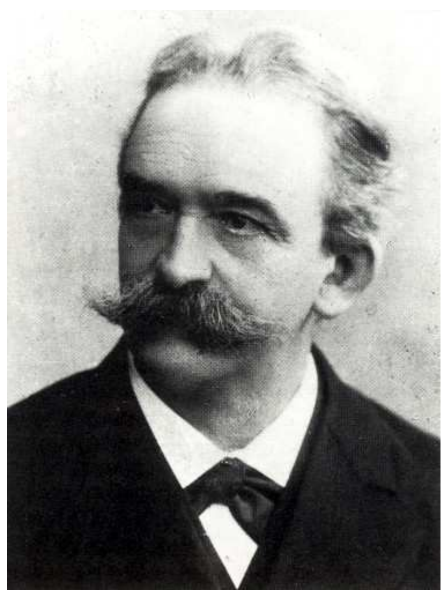
From (11.39a) and (11.39b) we find ***Mohr-Maxwell's formula*** (neglecting the factor ½), which was first derived by J.C.Maxwell in 1864 and applied in design practice in 1874 by O.Mohr, being called also *the unit-load method for linearly elastic structures:*

$$d_i = \sum \left[\int \frac{N \cdot n_i}{EA} dx + \int \frac{M \cdot m_i}{EI} dx + \int \frac{V \cdot v_i}{G\bar{A}} dx + \int \frac{M_t \cdot m_{ti}}{GI_p} dx \right] \qquad (11.40)$$



James Clerk Maxwell (1831-1879)

Relation (11.40) permits the computing of the displacement d_i (which can be an axial displacement Δ, a deflection v or w, or a rotation φ) in a section i on the direction of the force P_i.



Christian Otto Mohr (1835-1918)

To calculate a displacement using the energetically formula (11.40) we have to follow the steps:

a. Determine the diagrams of internal forces and moments N, V, M and M_t in structure, due to external real loads
b. Place a virtual unit load $\left(P_i = \bar{1}\right)$ in the section *i* where the displacement d_i is to be found. If d_i represent an axial displacement Δ_i or a deflection v_i or w_i, we place an unit virtual force in section *i* . If d_i represent a rotation φ_i, we place an unit virtual moment in section *i* .
c. Determine the diagrams n_i, m_i, v_i, m_{ti} due to this unit load $P_i = \bar{1}$
d. Introduce the terms in equation (11.40), integrate each term and finally summarize them

The integrals which intervene in *Mohr-Maxwell's formula* present the feature that the influence coefficients n_i, m_i, v_i, m_{ti} under the integral, have a linear variation. This is a consequence of the fact that they result from the application on the straight bar of a *unit virtual concentrated force or moment.* For this reason these integrals can be easily solved using the Vereshciaghin's integration rule, presented bellow.
All integrals having identical structure, we shall refer to the first term from equation (11.40), relative to bending:

$$J = \int \frac{M \cdot m}{EI} dx = \frac{1}{E} \int \frac{M}{I} m \cdot dx \qquad (11.41a)$$

For bars with constant cross section (I=const.) and the flexural rigidity EI=const.:

$$EI \cdot J = \int M \cdot m \cdot dx \qquad (11.41b)$$

We graphically represent the diagram M=M(x) in an orthogonal system of axis *xOM* (Fig.11.9a). Under this diagram another system of axis *xOm* is taken (Fig.11.9b), representing the linear function *m* , which makes an angle α with the abscissa *Ox.* The origin of both system of axis *xOM* and *xOm* is point *O* where the linear function *m* intersect the *x* axis (Fig.11.9b). In M diagram a differential element of area $d\Omega$ can be written: $d\Omega = M \cdot dx$. In the same *x* section in *m* diagram the corresponding value is: $m = x \cdot tg\alpha$. Replacing these in (11.41b) the integral is:

$$EI \cdot J = \int m(M \cdot dx) = \int x \cdot tg\alpha \cdot d\Omega = tg\alpha \int x \cdot d\Omega \qquad (11.42a)$$

The last integral is a static moment:

$$\int x \cdot d\Omega = \Omega \cdot x_G \qquad (11.42b)$$

where: Ω: is the area of *M* diagram
x_G: is the abscissa of this *M* diagram centroid

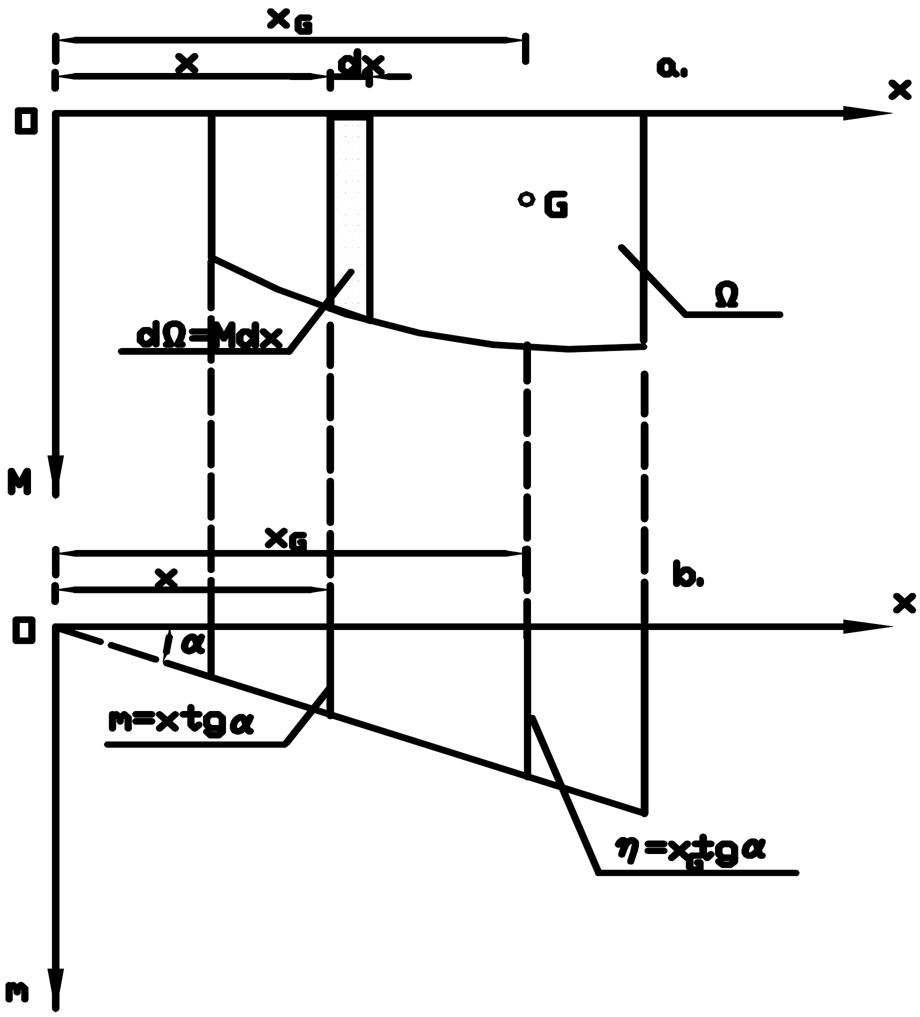


Fig.11.9

Replacing in (11.42a):

$$EI \cdot J = tg\alpha \cdot \Omega \cdot x_G = \Omega \cdot \eta \tag{11.42c}$$

Finally, the integral J is:

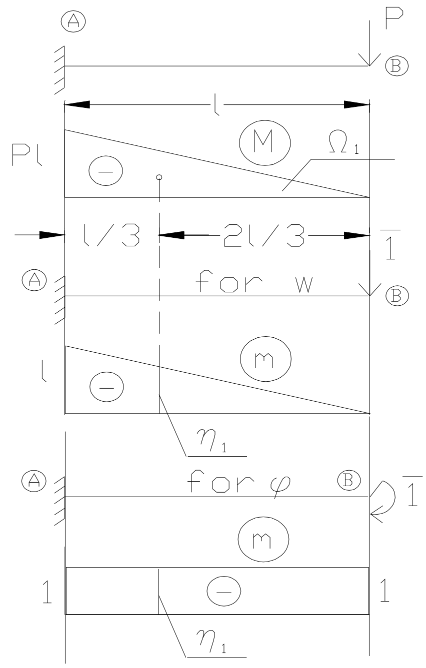
$$J = \int \frac{M \cdot m}{EI} dx = \frac{\Omega \cdot \eta}{EI} \tag{11.43}$$

Generally, from (11.43) we conclude that in order to calculate a displacement, first we have to calculate the area Ω of the real internal forces or moments diagrams (N, V, M, or M_t) and than these areas have to be multiplied by the ordinate η of the virtual internal forces or moments diagrams (n, v, m or m_t), corresponding to the centroid of Ω diagram. If the real internal forces or moments diagrams (N, V, M, or M_t) have also linear variation, the role of the two diagrams may be inverted.

11.7. MOHR-MAXWELL'S FORMULA FOR CALCULATION OF DISPLACEMENTS

11.7.1 We shall exemplify the method for the same examples illustrated in the chapter 7 (paragraphs 7.4.2 and 7.4.3):

For the cantilever loaded by the concentrate force P acting in the free end:



For the maximum deflection:

$$\Omega_1 = -\frac{Pl \cdot l}{2} = -\frac{Pl^2}{2}$$

$$\eta_1 = -\frac{2}{3}l$$

$$w_{max} = \frac{\Omega_1 \cdot \eta_1}{EI} = \frac{(-\frac{Pl^2}{2})(-\frac{2}{3}l)}{EI} = \frac{Pl^3}{3EI}$$

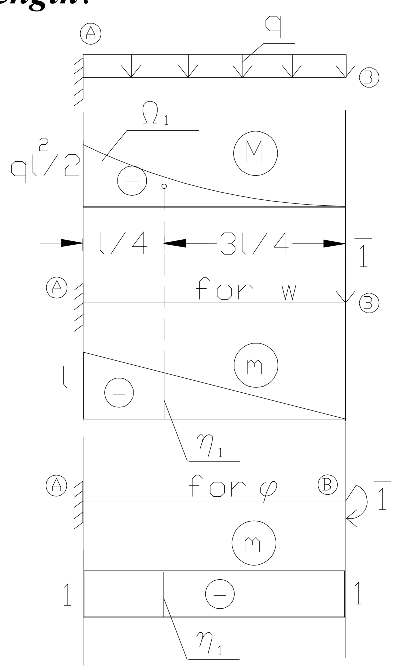
For the maximum rotation:

$$\Omega_1 = -\frac{Pl \cdot l}{2} = -\frac{Pl^2}{2}$$

$$\eta_1 = -1$$

$$\varphi_{max} = \frac{\Omega_1 \cdot \eta_1}{EI} = \frac{(-\frac{Pl^2}{2})(-1)}{EI} = \frac{Pl^2}{2EI}$$

For the cantilever loaded by the uniformly distributed force q on the entire length:



For the maximum deflection:

$$\Omega_1 = -\frac{ql^2/2 \cdot l}{3} = -\frac{ql^3}{6}$$

$$\eta_1 = -\frac{3}{4}l$$

$$w_{max} = \frac{\Omega_1 \cdot \eta_1}{EI} = \frac{(-\frac{ql^3}{6})(-\frac{3}{4}l)}{EI} = \frac{ql^4}{8EI}$$

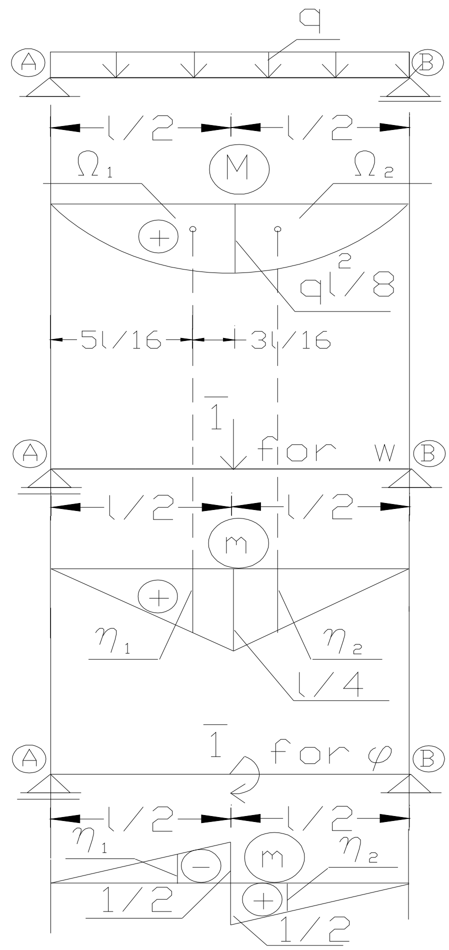
For the maximum rotation:

$$\Omega_1 = -\frac{ql^2/2 \cdot l}{3} = -\frac{ql^3}{6}$$

$$\eta_1 = -1$$

$$\varphi_{max} = \frac{\Omega_1 \cdot \eta_1}{EI} = \frac{(-\frac{ql^3}{6})(-1)}{EI} = \frac{ql^3}{6EI}$$

For the simple supported beam loaded by the uniformly distributed force q on the entire length:



For the maximum deflection:

$$\Omega_1 = \frac{2(\frac{ql^2}{8}\cdot\frac{l}{2})}{3} = \frac{ql^3}{24} = \Omega_2$$

$$\eta_1 = \frac{5}{8}\cdot\frac{l}{4} = \frac{5l}{32} = \eta_2$$

$$w_{max} = \frac{\Omega_1\cdot\eta_1 + \Omega_2\cdot\eta_2}{EI} = \frac{(\frac{ql^3}{24})(\frac{5l}{32})\cdot 2}{EI}$$

$$= \frac{5ql^4}{384EI}$$

For the rotation from the middle span:

$$\Omega_1 = \frac{2(\frac{ql^2}{8}\cdot\frac{l}{2})}{3} = \frac{ql^3}{24} = \Omega_2$$

$$\eta_1 = -\frac{5}{8}\cdot\frac{1}{2} = -\frac{5}{16}$$

$$\eta_2 = \frac{5}{8}\cdot\frac{1}{2} = \frac{5}{16}$$

$$\varphi = \frac{\Omega_1\cdot\eta_1 + \Omega_2\cdot\eta_2}{EI} = 0$$

11.7.1 For the following simple supported beam with a free end calculate the deflections w_1 and w_2 and the rotations φ_1 and φ_2 , assuming the second moment of area is $I = 20000 cm^4$ and the Young modulus is $E = 2.1\times10^6\ daN/cm^2$.

$$\Omega_1 = \frac{2(80\cdot 2)}{3} = \frac{320}{3} = \Omega_2$$

$$\Omega_3 = -\frac{135\cdot 2}{2} = -135 = \Omega_5$$

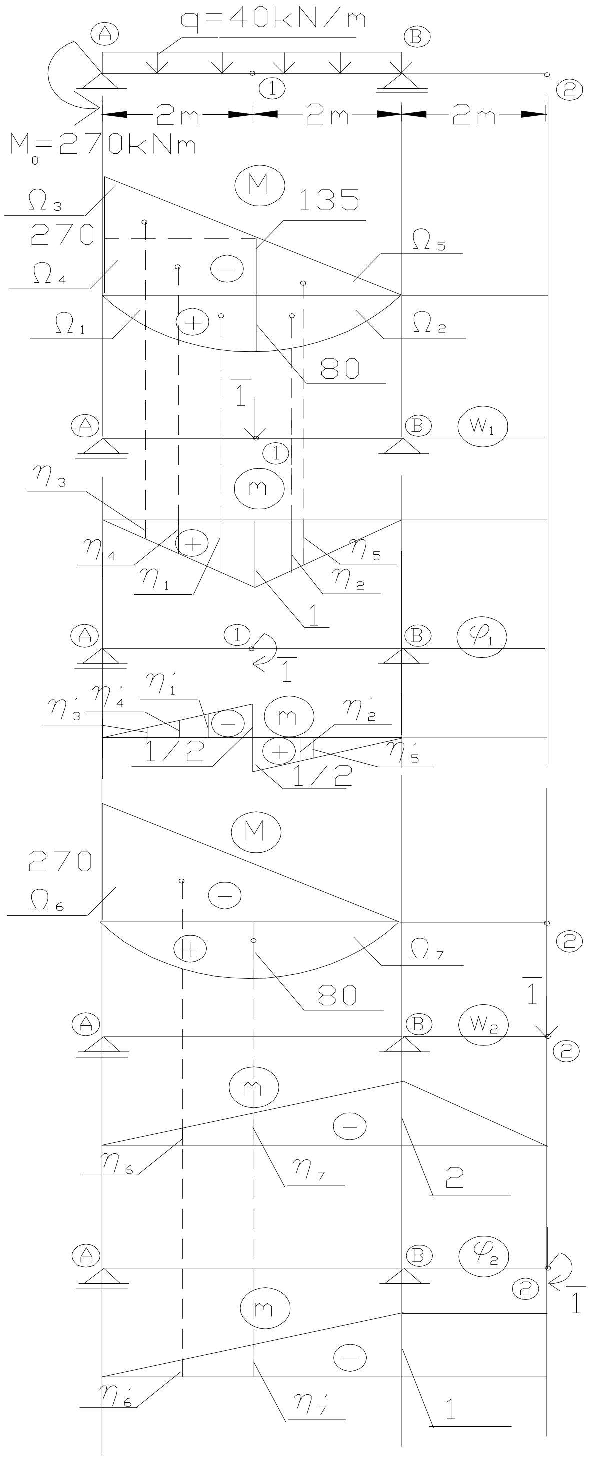
$$\Omega_4 = -135\cdot 2 = -270$$

$$\eta_1 = \frac{5}{8}\cdot 1 = \frac{5}{8} = \eta_2$$

$$\eta_3 = \frac{1}{3}\cdot 1 = \frac{1}{3}$$

$$\eta_4 = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$\eta_5 = \frac{2}{3}\cdot 1 = \frac{2}{3}$$



$$w_1=\frac{\left(\frac{320}{3}\right)\left(\frac{5}{8}\right)\cdot 2+(-135)\left(\frac{1}{3}\right)+}{EI}$$

$$\frac{+(-270)\left(\frac{1}{2}\right)+(-135)\left(\frac{2}{3}\right)}{}=$$

$$=\frac{-136.67\times 10^{8}}{2.1\times 10^{6}\cdot 20000}$$

$$=-0.325cm$$

$$\eta_1'=-\frac{5}{8}\cdot\frac{1}{2}=-\frac{5}{16};\eta_2'=\frac{5}{16}$$

$$\eta_3'=-\frac{1}{3}\cdot\frac{1}{2}=-\frac{1}{6}$$

$$\eta_4'=-\frac{1}{2}\cdot\frac{1}{2}=-\frac{1}{4}$$

$$\eta_5=\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{3}$$

$$\varphi_1=\frac{45}{EI}=\frac{45\times 10^{6}}{2.1\times 10^{6}\cdot 20000}=$$

$$=1.0714\times 10^{-3}\text{rad.}$$

$$\Omega_6=-\frac{270\cdot 4}{2}=-540$$

$$\Omega_7=\frac{2(80\cdot 4)}{3}=\frac{640}{3}$$

$$\eta_6=-\frac{1}{3}\cdot 2=-\frac{2}{3}$$

$$\eta_7=-\frac{1}{2}\cdot 2=-1$$

$$w_2=\frac{146.67}{EI}=\frac{146.67\times 10^{8}}{2.1\times 10^{6}\cdot 20000}$$

$$=0.349cm$$

$$\eta_6'=-\frac{1}{3}\cdot 1=-\frac{1}{3}$$

$$\eta_7'=-\frac{1}{2}\cdot 1=-\frac{1}{2}$$

$$\varphi_2=\frac{73.33}{EI}=\frac{73.33\times 10^{6}}{2.1\times 10^{6}\cdot 20000}=$$

$$=1.746\times 10^{-3}\text{rad.}$$