Chapter 9
BIAXIAL SHEARING

9.1 DEFINITION

As we have seen in the previous chapter, *biaxial (oblique) shearing* produced by the shear forces V_z and V_y, appears in a bar only accompanied by biaxial bending (we may discuss about pure oblique shearing only eventual in a cross section, but not in the entire bar). For this reason the types of loading which produce oblique shearing are the one presented in chapter 8 (Fig.8.2 and Fig.8.3):

a. The loads applied parallel to *Gz* axis will produce the shear force V_z (Fig.8.2), respectively those parallel to *Gy* axis will produce the force V_y.

b. The resultant shear force V from section (Fig.8.3) will be decomposed into the components:

$V_z = V\cos\alpha$ and $V_y = V\sin\alpha$

So, in both case of loading, each shear force will generate on turn shear (tangential) stresses τ_{xz} in the vertical elements (the webs) of the cross section and τ_{xy} in the horizontal elements (the flanges) of the cross section:

$$\mathrm{V_z}: \quad \tau_{xz}^{V_z} = \frac{V_z \cdot S_y(z)}{b_z \cdot I_y} \; ; \; \tau_{xy}^{V_z} = \frac{V_z \cdot S_y(\varsigma)}{b_\varsigma \cdot I_y}$$

$$\mathrm{V_z}: \quad \tau_{xz}^{V_y} = \frac{V_y \cdot S_z(\eta)}{b_\eta \cdot I_z} \; ; \; \tau_{xy}^{V_y} = \frac{V_y \cdot S_z(y)}{b_y \cdot I_z}$$

and finally, from superposing:

$$\tau_{xz} = \tau_{xz}^{V_z} + \tau_{xz}^{V_y}$$

$$\tau_{xy} = \tau_{xy}^{V_z} + \tau_{xy}^{V_y}$$

9.2 CROSS SECTIONS SYMMETRIC WITH RESPECT TO 2 PERPENDICULAR AXES (PRINCIPAL AXES)

9.2.1 Rectangular cross section

As we have seen at straight shearing (Chapter 8) the shear force V_z produced a tangential stress τ_{xz} having a parabolic distribution along the rectangle side which is parallel to the shear force V_z. Corresponding to this observation, the shear force V_y will produce a shear stress τ_{xy} having also a parabolic distribution, but now, along the rectangle side that is parallel to the shear force V_y.

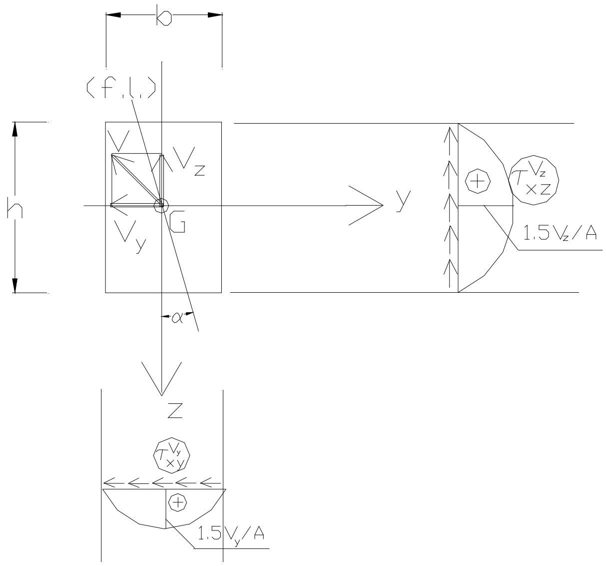


Fig.9.1

As we see in figures 9.1 and 9.2, the tangential stress τ_{xz} along the rectangle side of length h has the maximum value along Gy axis:

$$\tau_{xz_{\max}}^{T_z} = 1{,}5\frac{V_z}{A}$$

while the tangential stress τ_{xy} has the maximum value along Gz axis:

$$\tau_{xy_{\max}}^{T_y} = 1{,}5\frac{V_y}{A}$$

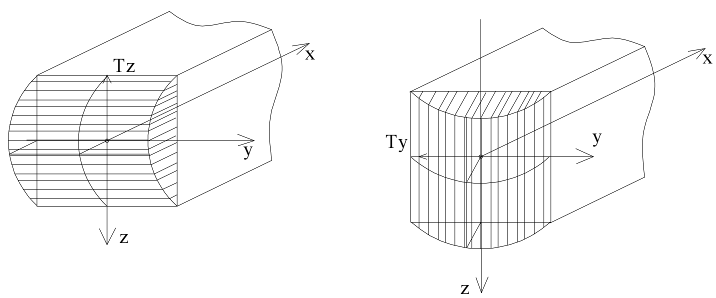


Fig.9.2

From the distribution of the tangential stresses presented in a perspective view in figure 9.2 we may conclude that the maximum values of both shear stresses $\tau_{xz\ max}$ and $\tau_{xy\ max}$ are superposed only in the cross section centroid, where:

$$\tau_{x_{\max}} = \sqrt{\tau_{xz}^2 + \tau_{xy}^2} = 1{,}5\frac{\sqrt{V_z^2 + V_y^2}}{A} = 1{,}5\frac{V}{A}$$

9.2.2 The double T (I) section, made from narrow rectangles

As we know from straight shearing, the shear force V_z will produce the shear stress τ_{xz} parallel to Gz axis, having a parabolic distribution and a maximum value in the neutral axis Gy (figure 9.3). From the same shear force, in both flanges shear stresses τ_{xy} will exist, parallel to Gy axis, having a linear distribution and maximum values in the web vicinity (Fig. 9.3).

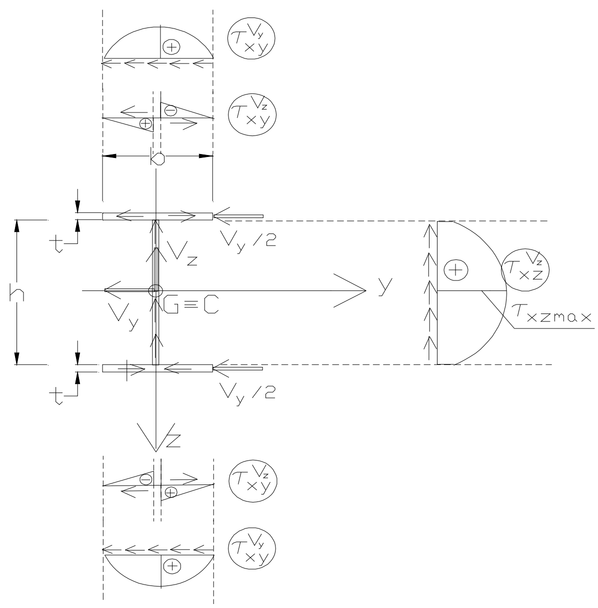


Fig.9.3

Similarly the shear force V_y will produce the shear stress τ_{xy} parallel to Gy axis (Fig. 9.3), with a parabolic distribution and a maximum value in the neutral axis Gz (practically, the maximum values of τ_{xy} are in the vicinity of the web, where the cross section width is much smaller, being the flange thickness). In the cross section web τ_{xz} produced by V_y will be zero (the static moment $S_z = 0$). For this reason the shear force V_y may be considered to be divided into 2 equal parts $V_y/2$ to each flange. We may admit a maximum value for the shear stress τ_{xy} produced by V_y corresponding to the narrow rectangle:

$$\tau_{xy_{\max}}^{V_y} = 1{,}5\frac{V_y}{A} = 1{,}5\frac{V_y}{2bt}$$

Finally, the maximum shear unit stresses are:

$\tau_{xz_{\max}} = \tau_{xz_{\max}}^{V_z}$: in the section corresponding to Gy axis

$\tau_{xy_{\max}} \cong \tau_{xy_{\max}}^{V_z} + \tau_{xy_{\max}}^{V_y}$: in the section corresponding to Gz axis

9.3 CROSS SECTIONS SYMMETRIC WITH RESPECT TO A SINGLE PRINCIPAL AXIS

9.3.1 Simple connex cross section

If the cross section is a monosymmetrical double T (I) section, the shear stresses distribution is similarly to the one presented in the previous paragraph, in figure 9.3. The difference is that the shear force V_y, which pass through the shear center C (whose position is unknown, yet), is now decomposed into 2 different components V_y^s and V_y^i (figure 9.4) inversely proportional to the distances h_s and h_i measured from C to the median lines of the flanges:

$$V_y^s = V_y \frac{h_i}{h_i + h_s} \text{ and } V_y^i = V_y \frac{h_s}{h_i + h_s}$$

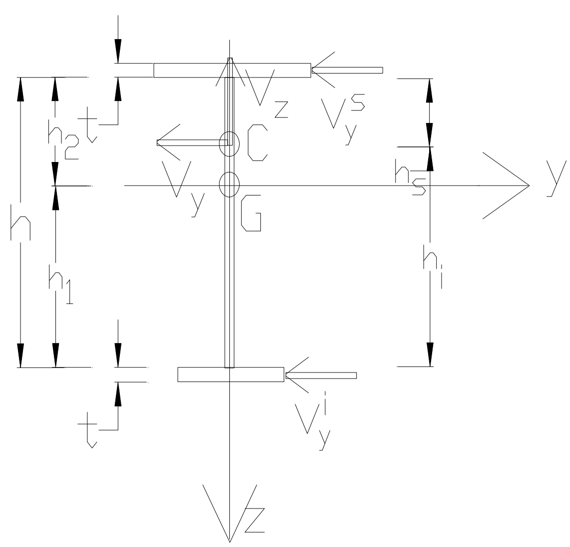


Fig.9.4

As this proceeding impose first the determination of the shear center position (presented in the next paragraph) we can't calculate the shear stresses τ from V_y as for the double symmetrical I section. We may calculate these shear stresses applying Juravski's formula:

$$\tau^{V_y} = \frac{V_y \cdot S_z}{b_y \cdot I_z}$$

In the above formula the static moment and the moment of inertia are written with respect to the neutral axis Gz.

Finally, the maximum shear unit stresses τ are calculated in the same manner adding the stress from V_z to the one from V_y.

If the monosymmetrical section has two webs (figure 9.5) we shall proceed in the same manner, writing Juravski's formula for τ, separately from each shear force V_z and V_y. The shear stresses distribution for this cross section is presented in Fig. 9.5.

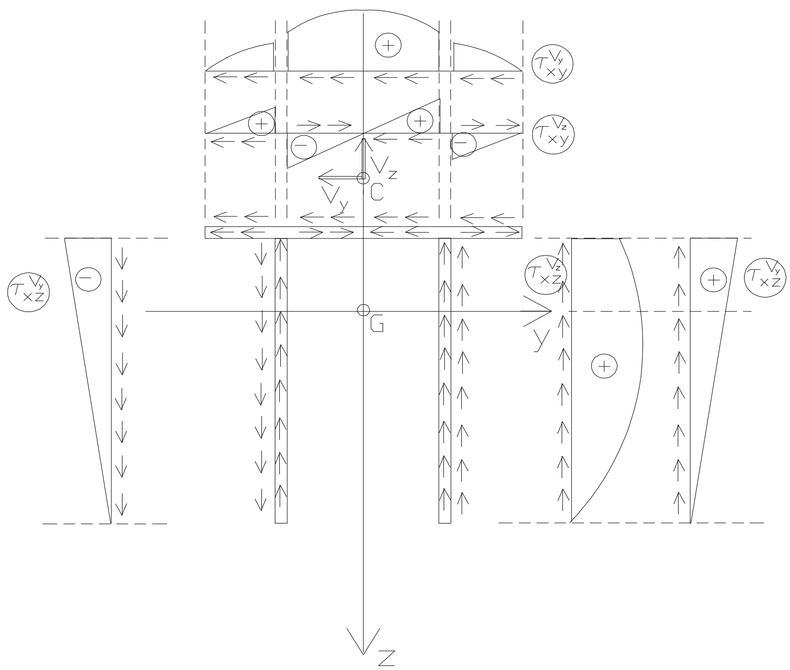


Fig.9.5

9.3.2 Sections of an arbitrary shape

In this paragraph we shall see an interesting section, presented in what concern the oblique shearing. This is *the angle with equal legs* (Fig. 9.6).

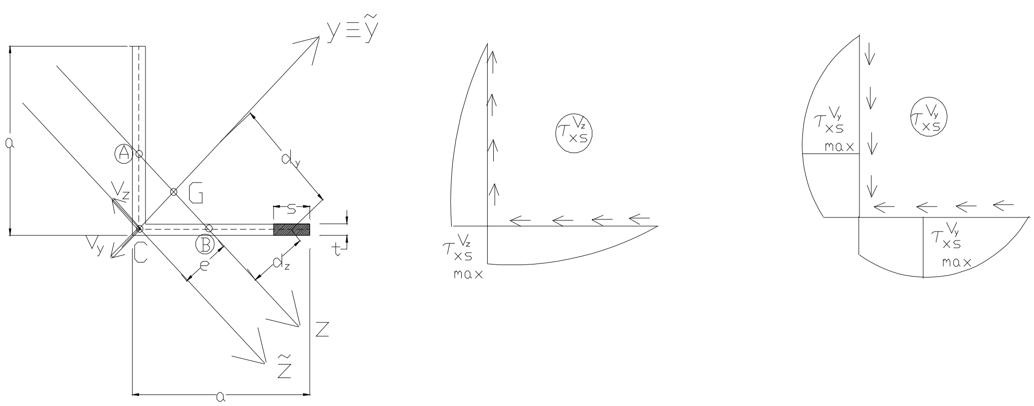
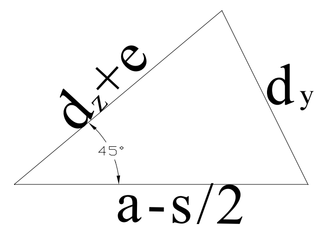


Fig.9.6

As Juravski's formula was demonstrated for the compound solicitation of shearing with bending, the formula is applied in the principal system of axis *yGz* of the cross section. So, whatever is the type of loading (fig.8.2 or 8.3), the shear forces will act in the shear center *C* (situated to the intersection of the median lines of each leg) parallel to *Gy* and *Gz* axis, the principal axis passing through the centroid *G*, but rotated with 45° with respect to the central system of axis $\overline{y}G\overline{z}$ (figure 9.6).

Now, the median lines aren't parallel with neither V_z nor V_y. That's why both shear stresses $\tau_x^{V_z}$ and $\tau_x^{V_y}$ will have a parabolic distribution along each leg. The maximum shear stress $\tau_{x\max}^{V_z}$ will be in the angle corner, as the neutral axis *Gy*, when calculating $\tau_x^{V_z}$, intersects the cross section median line in its corner. Similarly to this, the maximum shear stress $\tau_{x\max}^{V_y}$ will be in the points of intersection (points A and B) the neutral axis *Gz* (necessary when calculating $\tau_x^{V_y}$), with the cross section median line (figure 9.6).

To prove the parabolic distribution of both shear stresses $\tau_x^{V_z}$ and $\tau_x^{V_y}$, we consider a calculus level *s*. The distances from the centroid of the hatched area (figure 9.6) of length *s* to the principal system of axis *yGz*, are:



$$d_y = \left(a - \frac{s}{2}\right)\frac{\sqrt{2}}{2} \qquad \text{and} \qquad d_z = \left(a - \frac{s}{2}\right)\frac{\sqrt{2}}{2} - e$$

The static moments with respect to each principal axis, are:

$$S_y = t \cdot s \cdot d_y = t \cdot s \cdot \left(a - \frac{s}{2}\right)\frac{\sqrt{2}}{2}$$

$$S_z = t \cdot s \cdot d_z = t \cdot s \cdot \left[\left(a - \frac{s}{2}\right)\frac{\sqrt{2}}{2} - e\right]$$

As we may see from the above expressions, the variable *s* is at the second power in both static moments, what prove the parabolic distribution of $\tau_x^{V_z}$ and $\tau_x^{V_y}$.

Finally, in a certain section *s* the compound shear stress will be:

$$\tau_x = \tau_x^{V_z} + \tau_x^{V_y} = \frac{V_z \cdot S_y}{t \cdot I_y} + \frac{V_y \cdot S_z}{t \cdot I_z}$$

9.4 THE POSITION OF THE SHEAR (TORSIONAL) CENTER

The *shear center C* is the significant point from the cross section, in which the shear stresses τ are reduced. Its exact position is yet unknown, what we know until now is that the shear center is located somewhere on the symmetry axis, for the monosymetrical sections (if the cross section has 2 symmetry axis the shear center C is in their intersection, so it is identically with the centroid G).

9.4.1 Channel (U) section

We consider a channel (U shape) section with thin walls (Fig. 9.7). We admit that the shear force V_z pass through the shear center C, so the section is subjected only to uniaxial shearing. The torsion moment is null:

$$\sum (M_t)_C = 0$$

We represent the shear stresses distribution in the cross section walls. As we know, from V_z we represent the typical parabolic distribution of τ_{xz} in the cross section web, while in flanges the linear distribution of τ_{xy} is drawn. Along each median line the resultants of these shear stresses, R_1 and R_2, are also represented (figure 9.7).

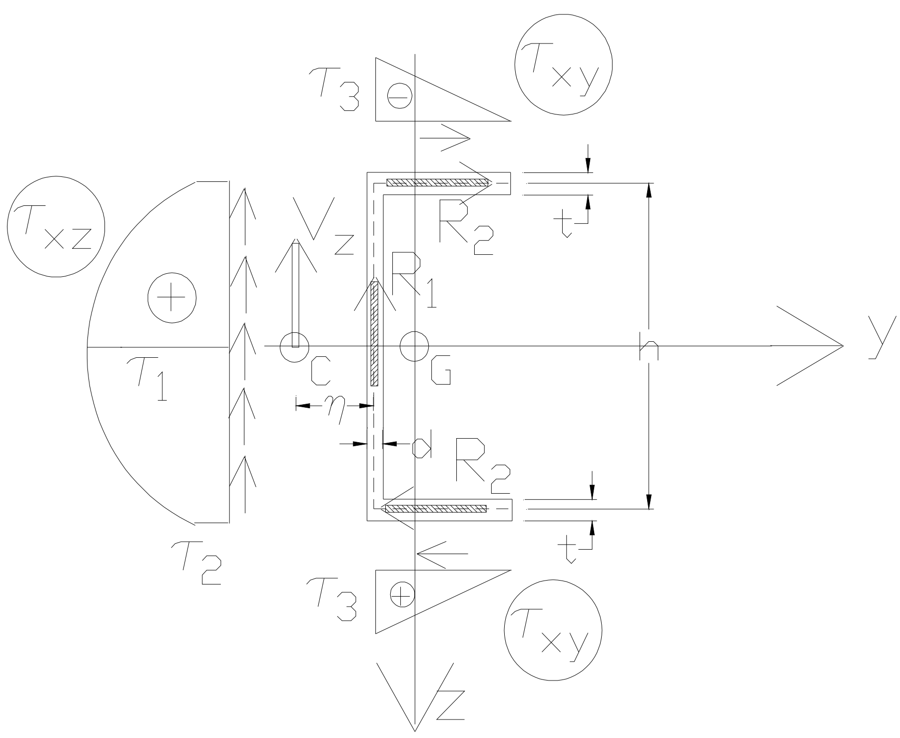


Fig.9.7

Assuming *Gy* is the neutral axis (the forces plan is parallel to *Gz*), the significant values of τ are:

$$\tau_{xz1} = \frac{V_z \cdot S_{y1}}{d \cdot I_y} = \frac{V_z}{d \cdot I_y}\left(bt \cdot \frac{h}{2} + \frac{h}{2} \cdot d \cdot \frac{h}{4}\right) = \frac{V_z}{dI_y}\left(\frac{bht}{2} + \frac{dh^2}{8}\right)$$

$$\tau_{xz2} = \frac{V_z \cdot S_{y2}}{d \cdot I_y} = \frac{V_z}{dI_y}\frac{bht}{2}$$

$$\tau_{xy3} = \frac{V_z \cdot S_{y3}}{t \cdot I_y} = \frac{V_z}{I_y}\frac{bh}{2}$$

The resultants of the shear stresses R_1 and R_2 are calculated as the volume of each τ diagram (taking account that, on the narrow rectangle thickness, τ is constant, having a uniform distribution):

$$R_i = A_\tau \cdot t_i$$

With this formula, each resultant is:

$$R_1 = \left[\tau_{xz2} \cdot h + \frac{2}{3}(\tau_{xz1} - \tau_{xz2})h\right]d = \frac{V_z}{I_y}\left(\frac{bh^2t}{2} + \frac{dh^3}{12}\right)$$

$$R_2 = \tau_{xy3} \cdot \frac{bt}{2} = \frac{V_z}{I_y}\frac{b^2ht}{4}$$

But:

$$I_y = \frac{dh^3}{12} + 2\frac{bt^3}{12} + 2bt\frac{h^2}{4} = \frac{dh^3}{12} + \frac{bh^2t}{2}$$

In the above expression of the moment of inertia I_y the second term is very small and it was neglected. This moment of inertia is replaced in the expression of R_1 and finally we obtain:

$$R_1 = V_z$$

We write the condition that the torsion moment with respect to the shear center *C*, is null:

$$\left(\sum M_t\right)_C = 0 : -R_1 \cdot \eta + R_2 \cdot h = 0 \Rightarrow \eta = \frac{R_2 \cdot h}{R_1}$$

After replacing R_1 and R_2 we obtain the distance η measured from the median line of the web to the shear center (Fig. 9.7):

$$\eta = \frac{b^2h^2t}{4I_y}$$

9.4.2 I section with unequal flanges

We want to compute the shear center position for the cross section presented in paragraph 9.3.1 (Fig.9.4). The shear center *C* is located on the symmetry axis *Gz*, to the distances h_s respectively h_i with respect to the median line of the superior respectively inferior flange (Fig.9.8).

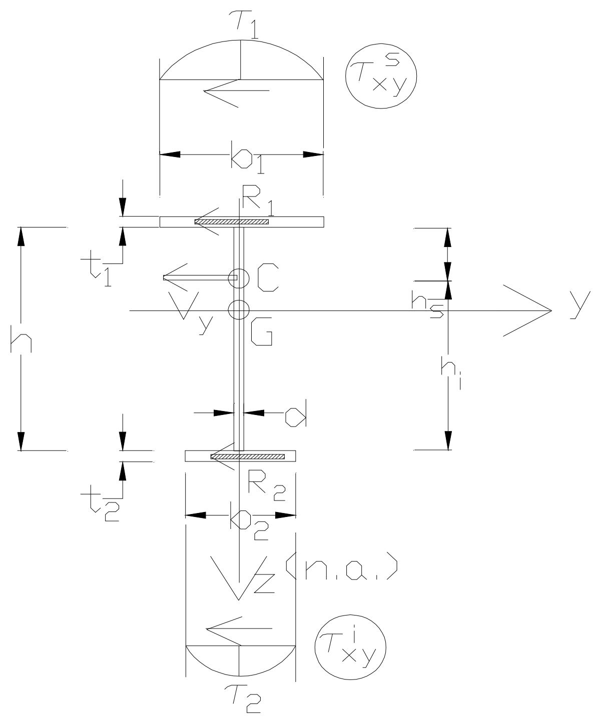


Fig.9.8

For straight bending with respect to *Gz* axis, which is the neutral axis, the corresponding shear force that produces the straight shearing is V_y. As we have seen in paragraph 9.3.1, V_y is decomposed in V_y^s and V_y^i, which give a parabolic distribution for τ_{xy} in each flange (Fig. 9.8). τ_{xz} from V_y in the cross section web is null (the static moment S_z=0).

The maximum values of τ_{xy} in each flange are:

$$\tau_{xy1} = \frac{V_y \cdot S_{z1}}{t_1 \cdot I_z} = \frac{V_y}{t_1 \cdot I_z}\left(\frac{b_1}{2} \cdot t_1 \cdot \frac{b_1}{4}\right) = \frac{V_y}{I_z}\frac{b_1^2}{8}$$

$$\tau_{xy2} = \frac{V_y}{I_z}\frac{b_2^2}{8}$$

The resultants R_1 and R_2 are fractile from V_y, respectively $V_y^s = R_1$ and $V_y^i = R_2$

$$R_1 = \frac{2}{3} \cdot \tau_{xy1} \cdot b_1 \cdot t_1 = \frac{V_y}{I_z} \cdot \frac{b_1^3 t_1}{12} = V_y \frac{I_{z1}}{I_z} = V_y^s$$

$$R_2 = \frac{2}{3} \cdot \tau_{xy2} \cdot b_2 \cdot t_2 = V_y \frac{I_{z2}}{I_z} = V_y^i$$

In the above relations I_{z1} and I_{z2} are the moments of inertia of each flange, with respect to the neutral axis Gz.

The torsion moment with respect to the shear center C, is null:

$$\left(\sum M_t\right)_C = 0 : -R_1 \cdot h_s + R_2 \cdot h_i = 0$$

But: $h_s + h_i = h \Rightarrow h_s = h - h_i$

Replacing h_s and R_1 and R_2 we get:

$$-\frac{V_y}{I_z} I_{z1}(h - h_i) + \frac{V_y}{I_z} I_{z2} h_i = 0 \quad \Rightarrow -I_{z1} h + (I_{z1} + I_{z2}) h_i = 0$$

But: $I_{z1} + I_{z2} \cong I_z \quad \left(I_{zw} = \frac{hd^3}{12} \cong 0 \right)$

Replacing in the above equation, we finally find the shear center position, through the distances:

$$h_i = h \frac{I_{z1}}{I_z}$$

And: $h_s = h \frac{I_{z2}}{I_z}$

9.4.3 The calculation steps used to determinate the position of the shear center C

We explain each step which should be followed in order to find the position of C, on the cross section with 2 webs from paragraph 9.3.1 (Fig. 9.9).

a. We position the shear center C on the ***symmetry axis*** (in our example Gz axis)
b. In C we apply a shear force perpendicular to the symmetry axis (V_y in our case)
c. From this shear force we establish the direction of the shear stresses flow in all narrow rectangles and we represent the diagrams of the shear stresses τ_x, τ_{xy} with parabolic distribution in the cross section flange, respectively τ_{xz} with linear variation in the cross section webs, but having different orientation and signs.
d. We compute all the significant values of the shear stresses τ_x from these diagrams. For our cross section we apply Juravski's formula

from V_y: $\tau_x = \dfrac{V_y \cdot S_z}{b_y \cdot I_z}$. As in all the values of τ_x the ratio V_y / I_z remain constant, we may note this ratio with k, so $k = V_y / I_z = cst.$, and all the significant values of τ_x will be function this constant : $\tau = f(k)$

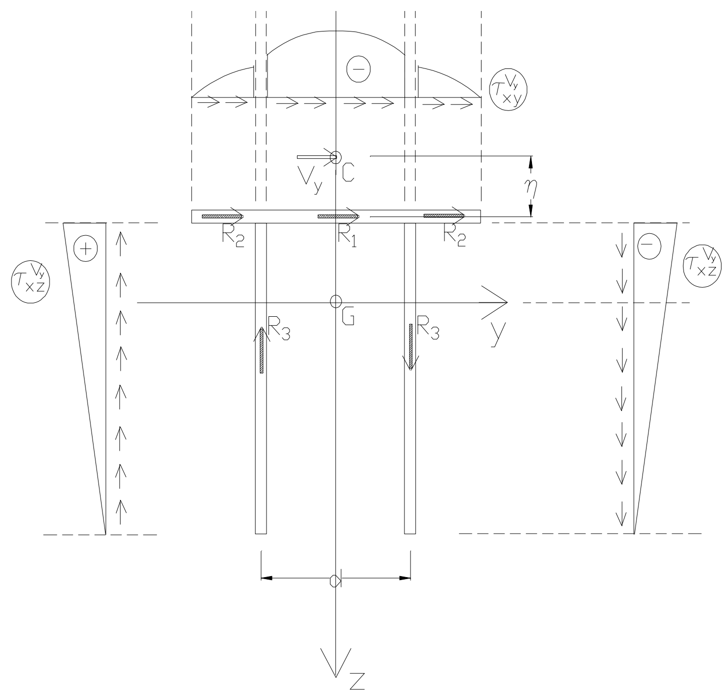


Fig.9.9

e. We compute the resultants R_i corresponding to each diagram τ_i represented on the narrow rectangle of thickness t_i. We apply the formula: $R_i = A_{\tau_i} \cdot t_i$, where $A_{\tau i}$ is the area of τ_i diagram.
f. We write the condition that the torsion moment with respect to the shear center C, is null. For our cross section (Fig. 9.9):
$$\left(\sum M_t\right)_C = 0 : -R_1 \cdot \eta - 2R_2 \cdot \eta + R_3 \cdot d = 0 \Rightarrow \eta = \frac{R_3 \cdot d}{R_1 + 2R_2}$$
g. From this condition we find the position of the shear center C given by this distance η.

If the cross section has no symmetry axis, the same steps must be followed, but separately on each axis, first from V_y with Gz the neutral axis and then from V_z with Gy the neutral axis. Finally we shall obtain 2 coordinates for the shear center C: η and ζ.

! All the calculations must be made in the principal system of axis yGz.

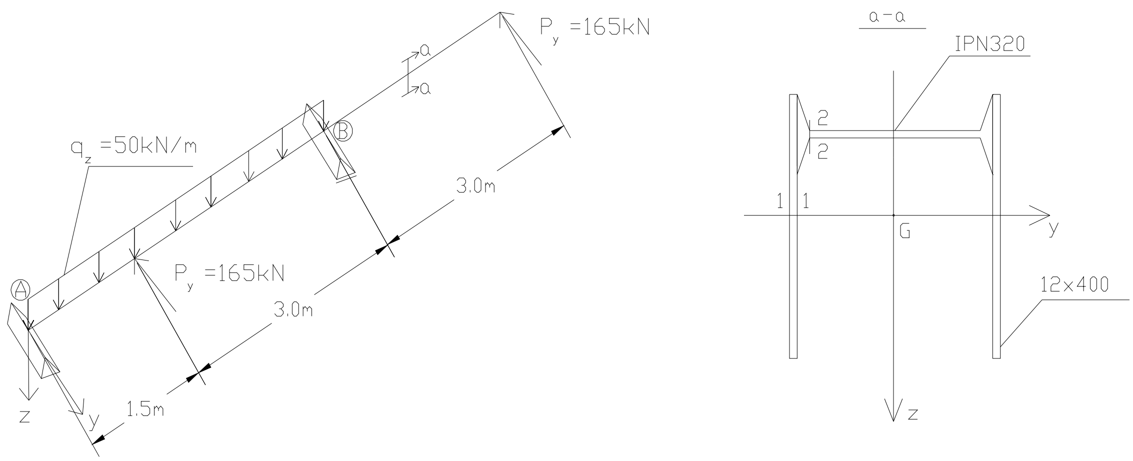
9.5 APPLICATION TO BIAXIAL BENDING WITH SHEARING

For the simple supported beam with the static scheme and the cross section from the figure bellow calculate:

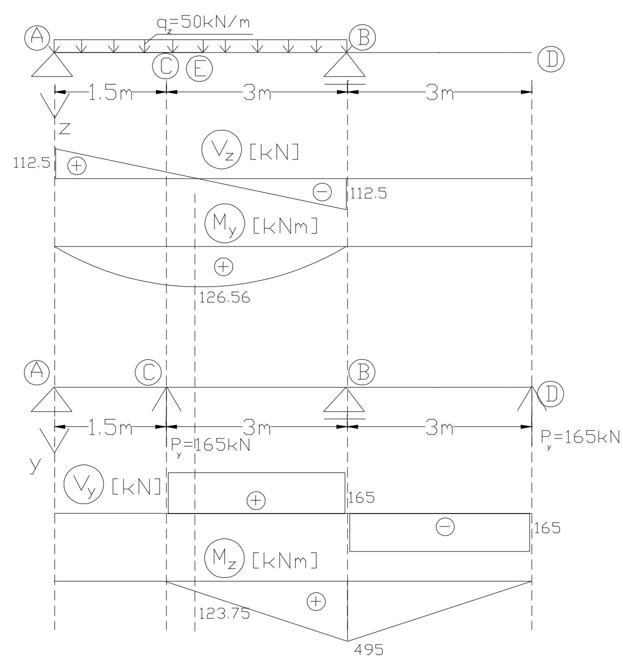
a. diagrams of stresses

b. the strength verification at bending ($\sigma_{x\,max} = 2200 daN/cm^2$). In the critical sections at bending the neutral axis and the normal stress diagram σ_x will be represented

c. in section B' (B left) the shear stress diagram τ_x with the resultant values at the levels 1-1 and 2-2



The diagrams of stresses:



The critical sections at bending are:

- section E, where $M_{y\,max} = 126.56$kNm and $M_z = 123.75$kNm, so a section of biaxial bending
- section B, where $M_y = 0$ and $M_{z\,max} = 495$kNm, so a section of uniaxial bending

With the second moments of area $I_y = 21129\text{cm}^4$ and $I_z = 38975\text{cm}^4$, the maximum normal stresses and the neutral axis are:

- in section E:

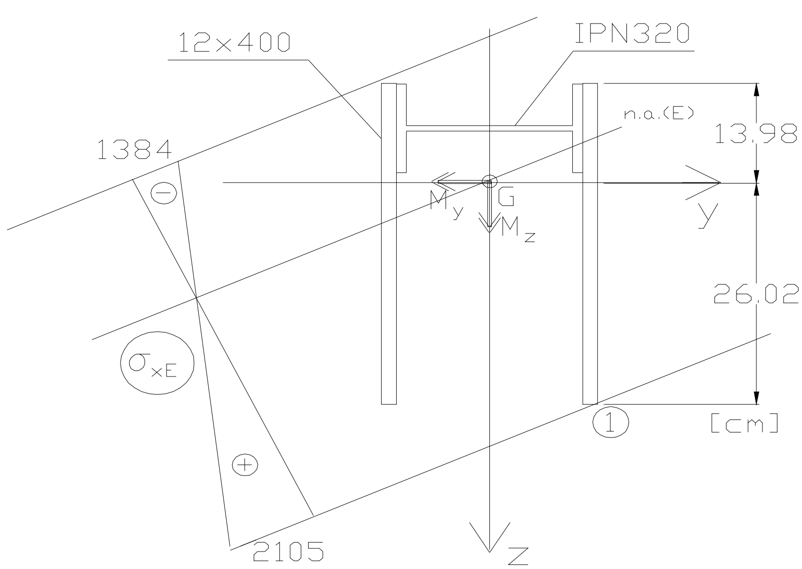
$$\sigma_{x\,max} = \frac{M_y}{I_y} z_1 + \frac{M_z}{I_z} y_1$$

$$= \frac{(126.56 \times 10^4)}{21129} 26.02 + \frac{(123.75 \times 10^4)}{38975} 17.2 =$$

$$= 2105 \frac{daN}{cm^2} < 2200 \frac{daN}{cm^2}$$

$$\sigma_{x\,min} = \frac{(126.56 \times 10^4)}{21129}(-13.98) + \frac{(123.75 \times 10^4)}{38975}(-17.2) =$$

$$= -1384 \frac{daN}{cm^2}$$

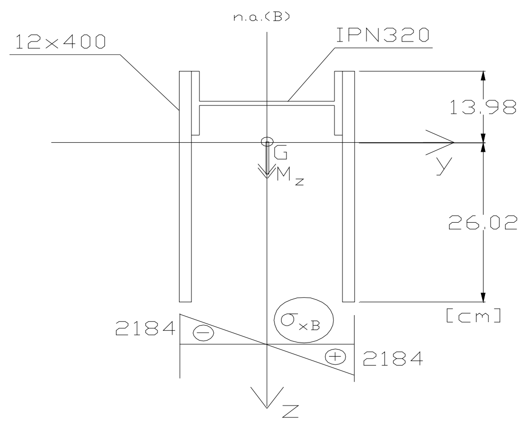


The neutral axis equation: $z = -\frac{M_z}{M_y}\frac{I_y}{I_z} y$

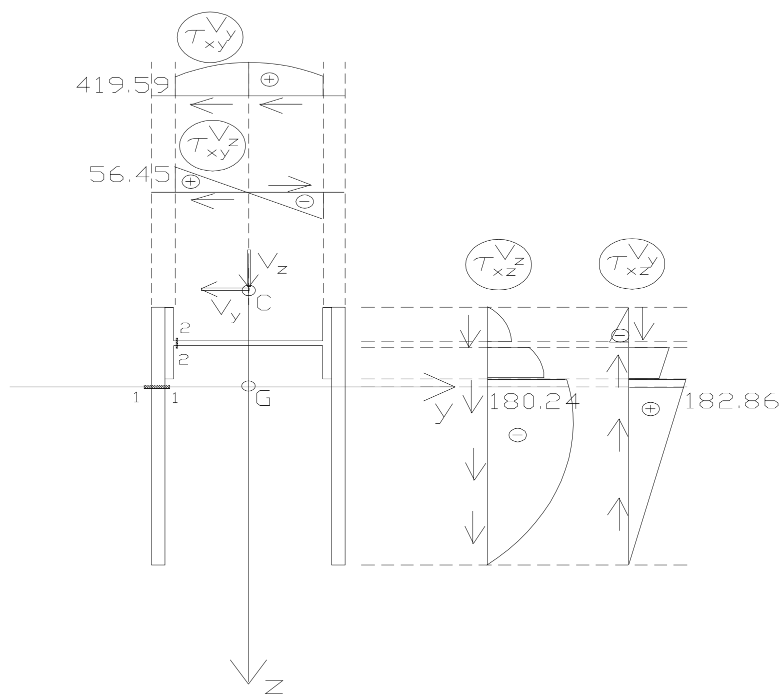
$z = -\frac{123.75}{126.56} \cdot \frac{21129}{38975} y$ → z = -0.53y

- in section B:

$$\sigma_{x\,max} = \frac{M_{z\,max}}{I_z} y = \frac{(495 \times 10^4)}{38975} 17.2 = 2184 \frac{daN}{cm^2} < 2200 \frac{daN}{cm^2}$$



In section B' the shear forces: $V_{z\,max}$ = -112.5kN and $V_{y\,max}$ = 165kN



$$\tau^{V_z}_{xz\,1-1} = -\frac{112.5 \cdot 10^2 (1.2 \cdot 26.02 \cdot 13.01)}{1.2 \cdot 21129} = -180.24 \frac{daN}{cm^2}$$

$$\tau^{V_y}_{xz\,1-1} = -\frac{165 \cdot 10^2 (1.2 \cdot 26.02 \cdot 16.6)}{1.2 \cdot 38975} = -182.86 \frac{daN}{cm^2}$$

$$\tau_{xz\,1-1} = -180.24 - 182.86 = -363.1 \frac{daN}{cm^2}$$

$$\tau_{\mathrm{xy}\ 2-2}^{V_z} = \frac{112.5\cdot 10^2(1.15\cdot 14.27\cdot 7.43)}{1.15\cdot 21129} = 56.45\,\frac{daN}{cm^2}$$

$$\tau_{\mathrm{xy}\ 2-2}^{V_y} = \frac{165\cdot 10^2(48\cdot 16.6+1.73\cdot 13.1\cdot 15.135)}{1.15\cdot 38975} = 419.59\,\frac{daN}{cm^2}$$

$$\tau_{\mathrm{xy}\ 2-2} = 56.45 + 419.59 = 476.04\,\frac{daN}{cm^2}$$