Chapter 8
BIAXIAL BENDING

8.1 DEFINITION

A cross section is subjected to *biaxial (oblique) bending* if the normal (direct) stresses σ_x from section are reduced to two bending moments M_y and M_z. Generally oblique bending is accompanied by oblique shearing, when the shear stresses τ_x are reduced to two shear forces V_z and V_y. If the shear forces $V_z = V_y = 0$, we discuss about *pure oblique bending*.

The positive convention of the vectors moment M_y and M_z is presented in Fig. 8.1. From figure we may conclude that the vectors moments M_y and M_z, which are included in the cross section plan, are positive if M_y acts inversely principal axis *Gy*, while M_z acts in the positive direction of principal axis *Gz*. In other words M_y and M_z are positive, when the resultant moment M_i stretches (tensions) the cross section first quadrant.

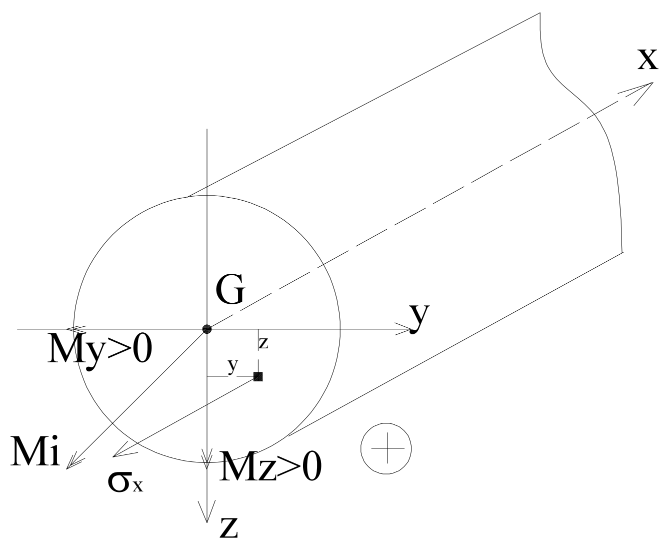

Fig.8.1

8.2 MODE OF LOADING

There are 2 main modes of loading which produce biaxial bending:

a. The loads are applied perpendicular to the torsion axis $C\tilde{x}$, in two plans which are parallel to the principal plans *xGy* and *xGz*.

We assume that for a certain cross section the systems of axis of the significant centers *G* and *C* are shown in Fig. 8.2.

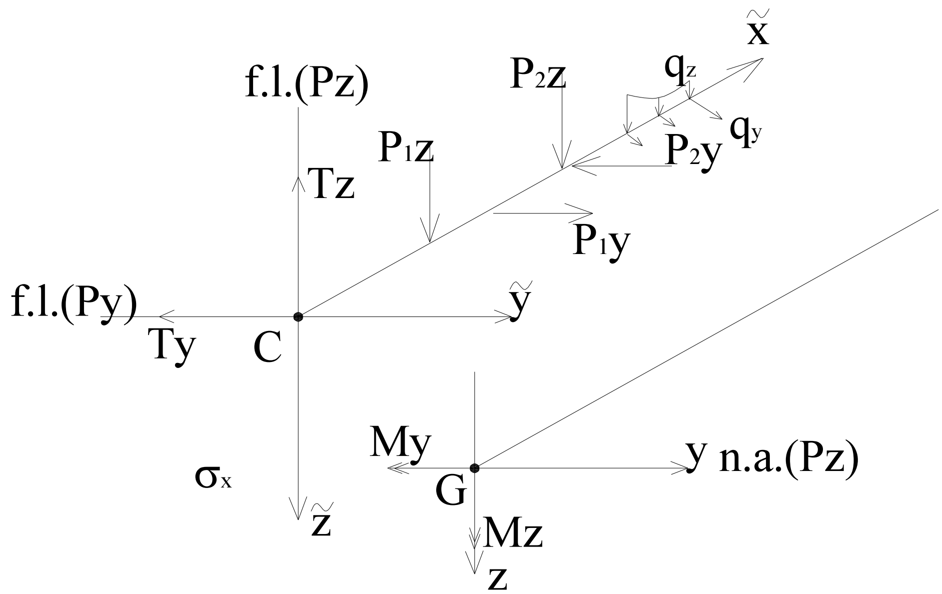

Fig. 8.2

! The forces lines must pass through the shear center C in order to have only oblique bending with shearing. Otherwise this compound solicitation is accompanied by torsion in that cross section.
From Fig. 8.2 we may conclude:
a.1 The vertical forces P_z will produce the shear force V_z acting in the shear center *C*, and the bending moment M_y acting in the centroid *G*.

The forces plan is $\tilde{x}C\tilde{z}$ and the forces line in any cross section is $C\tilde{z}$ axis.
The neutral axis for bending is *Gy* axis.
a.2 The vertical forces P_y will produce the shear force V_y acting in the shear center *C*, and the bending moment M_z acting in the centroid *G*.

The forces plan is $\tilde{x}C\tilde{y}$ and the forces line in any cross section is $C\tilde{y}$ axis.
The neutral axis for bending is *Gz* axis.

b. All forces act in a plan that passes through the shear center *C* (which contains $\tilde{x}$ axis), but it is inclined with respect to the principal plans *xGy* and *xGz* with an angle α (Fig. 8.3).

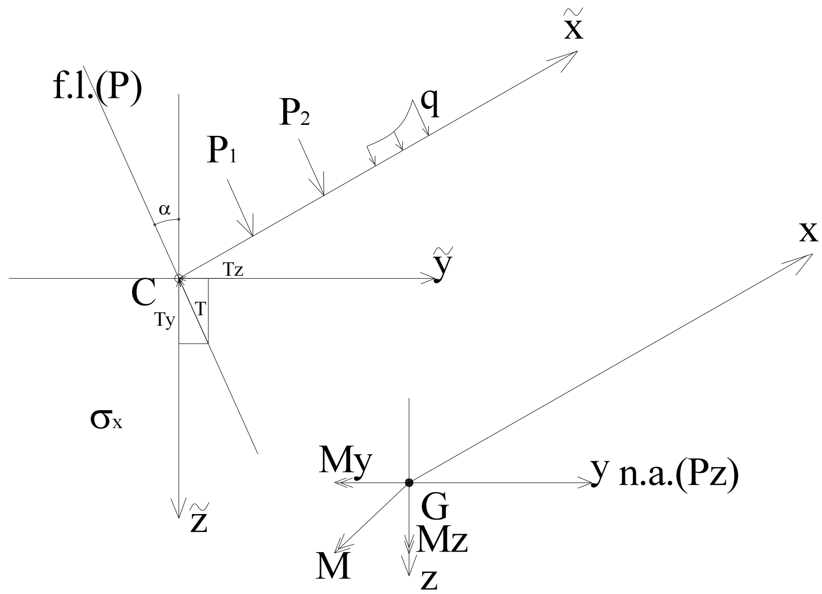

Fig. 8.3

The forces plan is inclined with the angle α, and the intersection of this plan with any cross section defines the forces line which is also inclined with the angle α with respect to $C\tilde{z}$ axis. Due to the forces acting in this inclined plan, shear forces *V* and bending moments *M* will subject any cross section.
This case of loading may be reduced to the first one (**a.**), decomposing from the beginning the forces *P* with respect to *Gy* and *Gz* axis:

$$P_z = P\cos\alpha \quad \text{and} \quad P_y = P\sin\alpha$$

As in the first case of loading, P_z will produce the shear force V_z and the bending moment M_y, while P_y will produce the shear force V_y and the bending moment M_z.
In the first case of loading the ratio M_y / M_z isn't constant along the bar, so that the deformed axis will be a skew curve in space. In the second case of loading the ratio M_y / M_z is constant in any cross section and the deformed axis will be a plane curve, but situated in a different plan with respect to the forces plan.

8.3 THE NORMAL STRESSES σ_x IN OBLIQUE BENDING

We consider a simple supported beam subjected to pure oblique bending, on the interval CD from this beam (Fig. 8.4). The forces *P*, inclined with an angle α with respect to *Gz* axis, produce between C and D sections only a bending moment *M* (without shear force), having a constant distribution on C-D interval. For simplicity we assume the cross section is symmetrical double, so $G \equiv C$ and the systems $xyz \equiv \tilde{x}\,\tilde{y}\,\tilde{z}$, are identically.

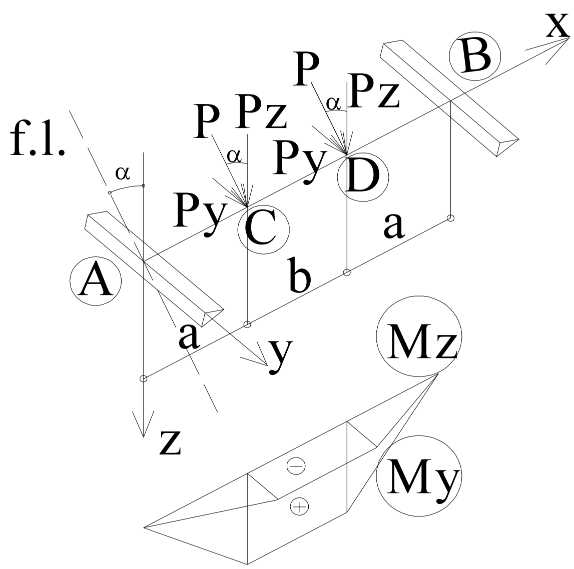

Fig.8.4

In cross section the plan of action the bending moment is A-A (Fig. 8.5), inclined with the angle α with respect to Gz axis. Also, the same angle is between the vector moment M and Gy axis (**Gy and Gz are principal axis**).

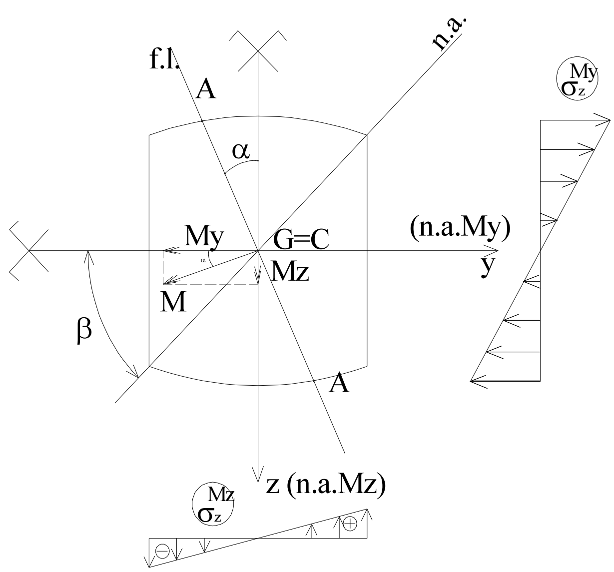

Fig.8.5

The components of the bending moment *M*, with respect to *Gy* and *Gz* axis, are:

$$M_y = M\cos\alpha \quad \text{and} \quad M_z = M\sin\alpha \tag{8.1}$$

They produce bending in plans *xGz*, respectively *xGy*. Each moment will generate normal stresses σ_x , calculated with Navier's formula:

$$\sigma_x^{M_y} = \frac{M_y}{I_y} z \quad \text{and} \quad \sigma_x^{M_z} = \frac{M_z}{I_z} y$$

Taking into account the hypothesis used in Mechanics of Materials, the hypothesis of the small deformations and the material has a linear elastic behaviour, the normal stresses σ_x may be calculated superposing the effects of the two straight bending (from M_y and M_z):

$\sigma_x = \sigma_x^{M_y} + \sigma_x^{M_z}$, or:

$$\boxed{\sigma_x = \frac{M_y}{I_y} z + \frac{M_z}{I_z} y} \tag{8.2}$$

The positive moments M_y and M_z are those which produce tensional stresses σ_x in any point from the first quadrant of the principal system of axis *yGz* (positive *y* and *z* coordinates).

Replacing M_y and M_z from (8.1) in the relation of σ_x (8.2), we get:

$$\sigma_x = M\left(\frac{z\cos\alpha}{I_y} + \frac{y\sin\alpha}{I_z}\right) \tag{8.3}$$

The neutral axis equation is obtained from the condition $\sigma_x = 0$. As, in (8.3) the bending moment $M \neq 0$, the single solution is:

$$\frac{z\cos\alpha}{I_y} + \frac{y\sin\alpha}{I_z} = 0 \text{ , or: } \boxed{z = -tg\alpha \frac{I_y}{I_z} y} \tag{8.4}$$

Replacing: $\frac{M_z}{M_y} = \frac{M\sin\alpha}{M\cos\alpha} = tg\alpha$, in (8.4), we obtain another form for the neutral axis equation:

$$\boxed{z = -\frac{M_z}{M_y}\frac{I_y}{I_z} y} \tag{8.5}$$

Equations (8.4) and (8.5) represent the equation of the neutral axis in case of oblique bending, which can be written shorter:

$$z = -m \cdot y \tag{8.6}$$

where *m*: represent the neutral axis slope

$$m = \frac{M_z}{M_y}\frac{I_y}{I_z} = tg\beta \tag{8.7}$$

Replacing (8.7) in equation (8.6), we obtain:

$$z = -tg\beta \cdot y \tag{8.8}$$

This final equation (8.8) shows that the neutral axis is a straight line passing through the centroid *G(0,0)* and being inclined with the angle β with respect to *Gy* axis.

! *Note that in case of oblique bending, the neutral axis does no more coincide with the support of the moment vector M* (*M* doesn't act around the neutral axis, because $tg\beta \neq tg\alpha \cdot \frac{I_y}{I_z}$). Exception to this observation, are the cross sections having the moments of inertia $I_y = I_z \Rightarrow tg\beta = tg\alpha$ (as for the square or circular cross sections).

! *As* $\beta \geq \alpha$*, the neutral axis is no more perpendicular to the forces line.*

To trace the neutral axis we either determine the angle β ($\beta = arctg\left(\frac{M_z}{M_y}\frac{I_y}{I_z}\right)$), or we define the second point *Q* (figure 8.6) through which the neutral axis will pass (the first point is the centroid *G*), of abscissa *1* and ordinate *–m*: *Q(1,-m).*

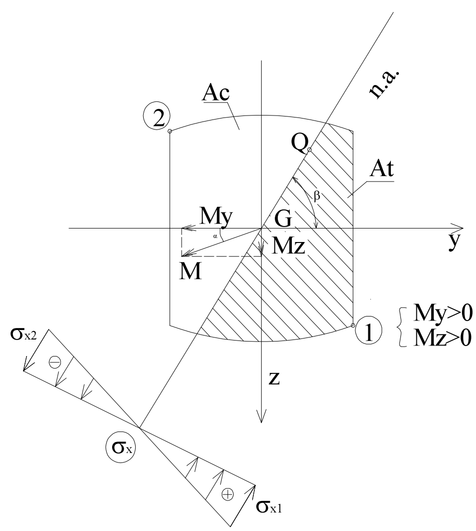

Fig.8.6

! *Note that the neutral axis passes always through the quadrant limited by the two vectors moment* M_y *and* M_z*.*

After tracing the neutral axis, two parallels to the neutral axis which are tangent to the cross section contour, are also traced. These two parallels (fig.8.6) will pass through the extreme points $1(y_1,z_1)$ and $2(y_2,z_2)$.

The normal (direct) stress σ_x diagram is drawn considering the reference line (which is even the cross section) perpendicular to the neutral axis and having extreme values, maximum tensile stress and maximum compressive stress, in point *1*, respectively *2* (fig.8.6):

$$\sigma_{x_1} = \frac{M_y}{I_y} z_1 + \frac{M_z}{I_z} y_1 > 0$$

$$\sigma_{x_2} = \frac{M_y}{I_y} z_2 + \frac{M_z}{I_z} y_2 < 0$$

As in practical calculations we have to check the strength capacity of the most subjected cross section, meaning that the strength condition $|\sigma_{x\max}| \leq R$ must be verified, we are interested mainly in computing the maximum normal stress (tension or compression, in case of steel cross sections):

$$\sigma_{x_{\max}} = \frac{M_{y\max}}{I_y} z_{\max} + \frac{M_{z\max}}{I_z} y_{\max} \leq R \qquad (8.9)$$

This means that we shall work in the critical (most dangerous) sections, where M_y and M_z have maximum values. If M_y and M_z aren't maximum in the same section we have to consider 2 critical sections:

Section 1: $M_{y\ max.}$ and $M_{z\ afferent}$

Section 2: $M_{y\ afferent}$ and $M_{z\ max.}$

The strength condition will be:

$$\sigma_{x_{\max}} = \max\left(\sigma_{x_1}, \left|\sigma_{x_2}\right|\right) \leq R \qquad (8.10)$$

In case of sections that may be inscribed in a rectangle (Fig.8.7) and if there is material in the corners of rectangle (ex: rectangle, double T(I) shaped cross sections, or any other composed section which form a rectangle) we may determine that normal stresses without determining first the neutral axis position. The maximum stresses $\sigma_{x\,max}$ will be, no doubt, in the cross section corners:

$$\sigma_{x\,max} = \pm \frac{M_y}{W_y} \pm \frac{M_z}{W_z}$$

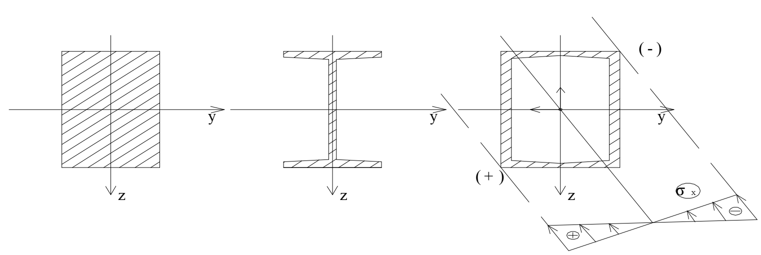

Fig.8.7