Chapter 7

UNIAXIAL BENDING WITH SHEARING IN STRAIGHT BARS

7.1 GENERALS

Bending of straight bars is the result of the action of transversal exterior forces and couples which produce bending moments in cross sections (Fig.7.1). As a consequence of this action, the longitudinal bar axis become curve. The bent bars are called **beams.**

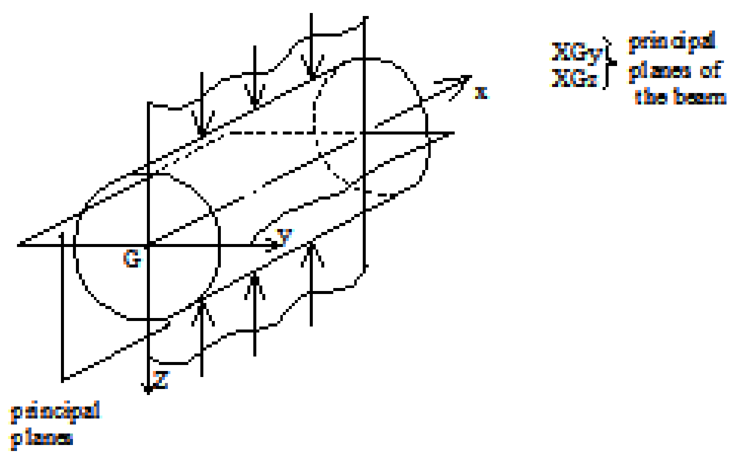


Fig.7.1

If the forces plan (*xGz* or/and *xGy*) contains the longitudinal bar axis *x*, reducing the stresses in the centroid G, we shall obtain the bending moments M_y, M_z and the shear forces V_z and V_y characteristic to a compound solicitation, represented in cross section with their positive convention (Fig.7.2).

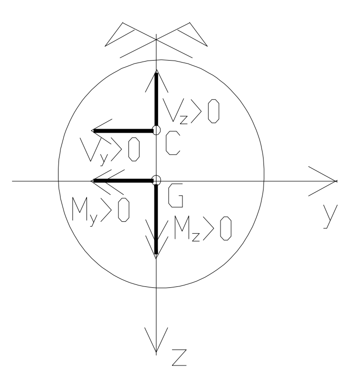


Fig.7.2

We may have simple or compound solicitations:

Uniaxial (straight) bending: only one moment acts in the cross section: M_y *or* M_z

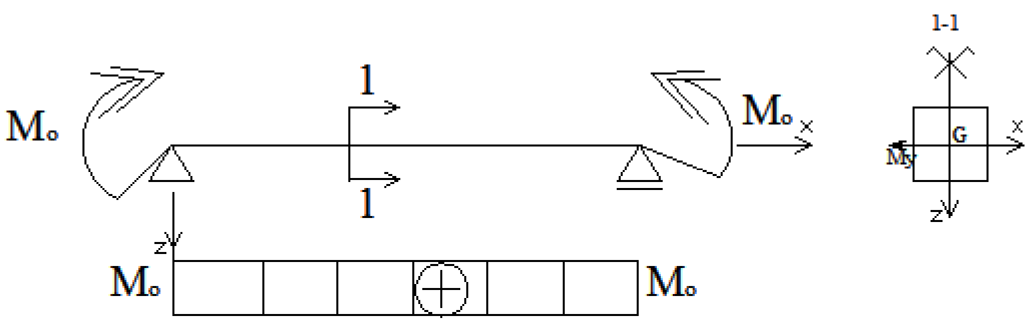
Biaxial (Oblique) bending: both moments M_y *and* M_z act in the cross section

Pure bending: when the shear forces in the cross section are missing: $V_z = V_y = 0$; it can be *straight* or *oblique pure bending*

Bending with shearing: when all stresses M_y *and/or* M_z , V_z *and/or* V_y act in the cross section; it can be *straight* or *oblique bending with shearing*

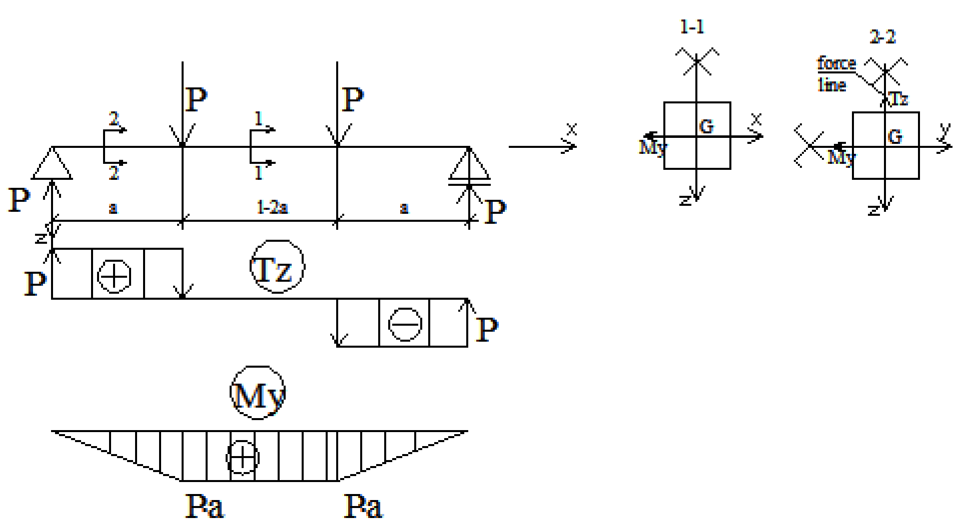
Examples:

a)



A prismatic member subjected to equal and opposite couples acting in the same longitudinal plane. On the entire length the member is subjected to ***pure bending***.

b)

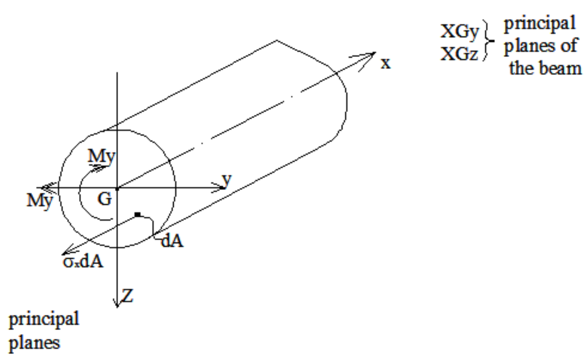


A prismatic member subjected by two equal point forces acting to an equal distance to supports. In this case, only in the central zone (between the loads P) ***pure bending*** occurs. On both length *"a'* from bar, ***straight bending with shearing*** subject the bar. The forces **P** act in the ***forces plan***, and the intersection of this plan with the cross section is called the ***force line*** (f.l.).

7.2 PURE BENDING

The previous example a) is the case of a straight bar subjected to ***pure bending*** (only the bending moment M_y acts in any cross section of the bar, the sum of the components of the forces in any direction is zero), ***but only if the bar self-weight is neglected.***

For a cross section subjected to pure bending (Fig.7.3), the stresses are:



$$N = \int_A \sigma_x dA = 0$$

$$M_y = \int_A \sigma_x dA \cdot z \neq 0$$

$$M_z = \int_A \sigma_x dA \cdot y = 0$$

$$V_z = V_y = 0$$

Fig.7.3

7.2.1. Geometrical aspect

We shall use the same rubber model as the one used for axial solicitation (centric tension). But in this case the prismatic model will be subjected as in example a) to equal and opposite couples M_0 (Fig.7.4).

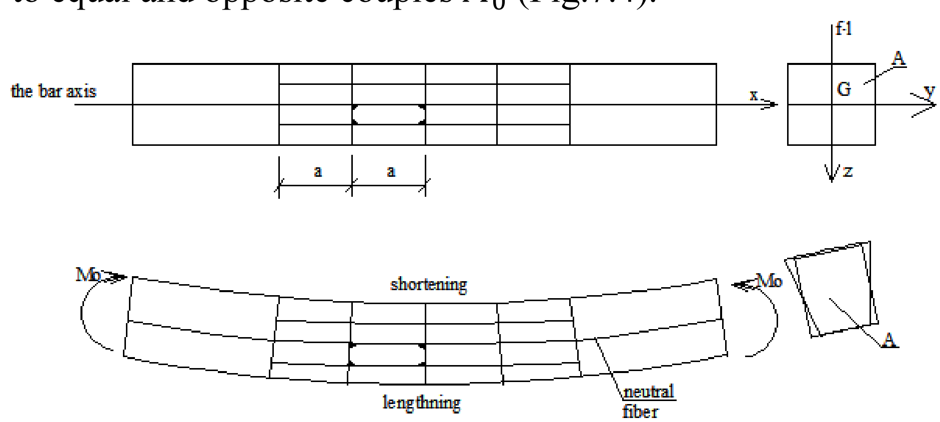


Fig.7.4

The same network of straight lines is traced on the lateral surface of the model. The lines are parallel and equidistant.

Admitting that, what we see on the lateral surface is valid in all the planes which are || to this surface, instead of vertical lines we discuss about cross sections and instead of longitudinal lines we discuss about longitudinal sections.

After deformation, the longitudinal sections initial straight become curve, they bend uniformly to form a circular arc, the length of the top part decreases and the length of the bottom part increases. The network remains still rectangular because the cross sections initial perpendicular to the longitudinal sections, remain plane and perpendicular to the curved longitudinal section (strips), so the hypothesis of Bernoulli is valid.

Passing from lengthening of the longitudinal strips to their shortening is made continuous on the model height. This means that it will be a longitudinal strip that even if is curved it doesn't change its length. It is a ***neutral surface*** that is parallel to the upper and lower surfaces and for which the length does not change. This strip is called ***neutral strip.*** The deformed axis of the bent beam is called ***deformed fiber*** or ***line.*** The intersection between the neutral strip and cross section is called ***neutral axis.***

We observe also that the initial straight angles of the network remain straight after deformation, what means that the specific sliding is null:

$$\gamma_{xz} = \gamma_{xy} = \gamma_{yz} = 0$$

In what concern the specific elongations ε_x, we isolate a differential element from the deformed beam, of length *dx* (Fig.7.5).

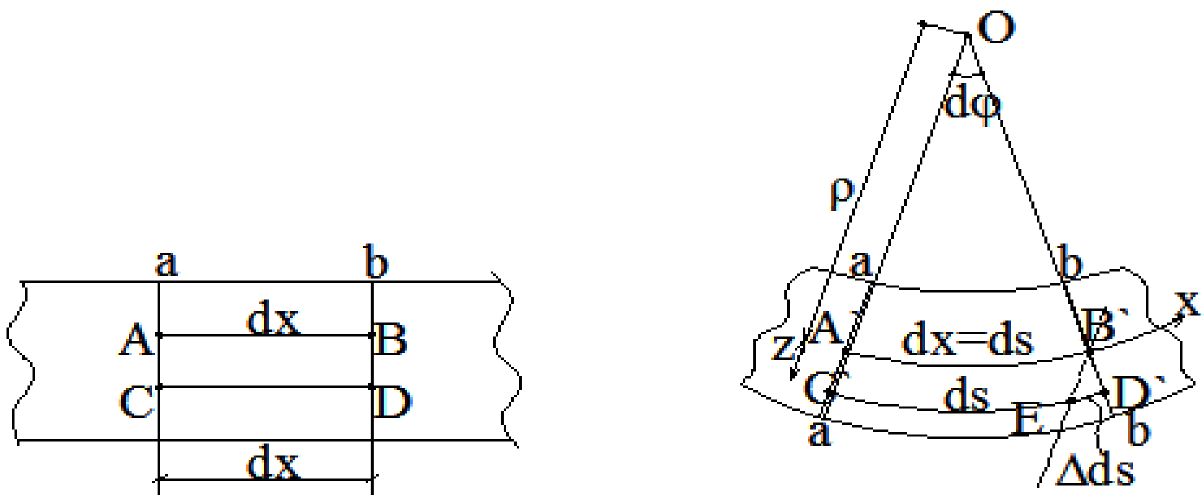


Fig.7.5

On figure 7.5 , ρ represents the radius of curvature of the neutral strip (constant on *dx*) and point *O* is the center of curvature.

The elementary sections *a-a, b-b* perpendicular to the longitudinal bar axis before deformation form an elementary angle $d\varphi$ between them, after deformation.

After deformation the neutral strip AB of length *dx* is curved, but it will have the same length AB = A'B'= dx = ds = ρ dφ.

At a level *z* the lengthening of a fiber can be written from the resemblance of the curvilinear triangles OA'B' and B'ED':

$$\frac{\rho}{z} = \frac{dx}{\Delta dx} \rightarrow \Delta dx = \frac{z \cdot dx}{\rho}$$

The specific elongation ε_x:

$$\varepsilon_x = \frac{\Delta dx}{dx} = \frac{z}{\rho} \qquad \boxed{\varepsilon_x = \frac{z}{\rho}}$$

As, in the above relation of ε_x the coordinate *z* appear at 1st power, this shows that the strain ε_x varies linearly on cross section, the stresses and strains being negative (compressive) above the neutral plane respectively positive (tension) below it .

The other two strains ε_y and ε_z are neglected:

$$\varepsilon_y = \varepsilon_z = -\mu\varepsilon_x = -\frac{\mu}{\rho}z \cong 0$$

7.2.2. Physical aspect

Admitting that the solicitation takes place in the linear elastic domain, the law of Hooke is valid:

$$\sigma = E \cdot \varepsilon \text{ and } \tau = G \cdot \gamma \rightarrow \tau_{xz} = \tau_{xy} = \tau_{yz} = 0$$

$$\boxed{\sigma_x = \frac{E \cdot z}{\rho} \neq 0}$$

$$\sigma_y = \sigma_z = 0$$

So, the single distinct unit stress is the normal stress σ_x which, similarly to the specific strain ε_x, varies linearly on the cross section height, being 0 in the neutral axis (n.a.) and maximum (tension and compression) in the extreme fibers. On the bar width, at any level *z*, σ_x is constant (uniformly distributed).

The relation of σ_x can't be yet used, because:

1. We don't know the radius of curvature ρ
2. We don't know the position of the neutral axis.

7.2.3. The static aspect

We write the stresses from a static calculus (from exterior) and from a strength calculus (from interior). For our solicitation the single stress is $M_y = M_0$, the other two (which produce σ_x) $N = M_z = 0$. These are the stresses from exterior (static). From interior we write the strength definition of these stresses:

1) $N = \int_A \sigma_x \, dA = \int_A \frac{Ez}{\rho} dA = \frac{E}{\rho} \int_A z \, dA = 0$

As $\frac{E}{\rho} \neq 0$ (ρ can't be infinite for $\frac{E}{\rho} = 0$, because this means that the bar remains straight) => $S_y = \int_A z \, dA = 0$ (a)

2) $M_z = \int_A \sigma_x \, y dA = \int_A \frac{Ez}{\rho} y dA = \frac{E}{\rho} \int_A yz \, dA = 0$

=> $I_{yz} = \int_A yz \, dA = 0$ (b)

Relation (a) shows that the first moment of area S_y with respect to the neutral plane is zero. Therefore, the neutral surface must pass through the section centroid, what means that *Gy* axis is a central axis.

Relation (b) shows that the centrifugal moment of inertia I_{yz} is null, so the system *yGz* is the principal system. ***From these two observations, we may conclude that yGz is the principal system of axes, with the origin in the centroid G and the neutral axis is Gy axis*** (when the force line is *Gz* axis).The vector of the bending moment M_y acts along this neutral axis *Gy*.

3) $M_y = \int_A \sigma_x \, z dA = \int_A \frac{Ez}{\rho} z dA = \frac{E}{\rho} \int_A z^2 \, dA = \frac{E}{\rho} I_y = M_0$

$\rho = \frac{E}{M_y} I_y \rightarrow \boxed{\frac{1}{\rho} = \frac{M_y}{EI_y}}$: defines *the curvature of deformed longitudinal axis*

Replacing the curvature in the relation of σ_x:

$\sigma_x = \frac{Ez}{\rho} = \frac{E\,z\,M_y}{EI_y}$

Finally we may write **the formula of Navier**:

$$\boxed{\sigma_x = \frac{M_y}{I_y} z} \qquad \left[\frac{daN}{cm^2}\right]$$

For a rectangular cross section σ_x diagram is:

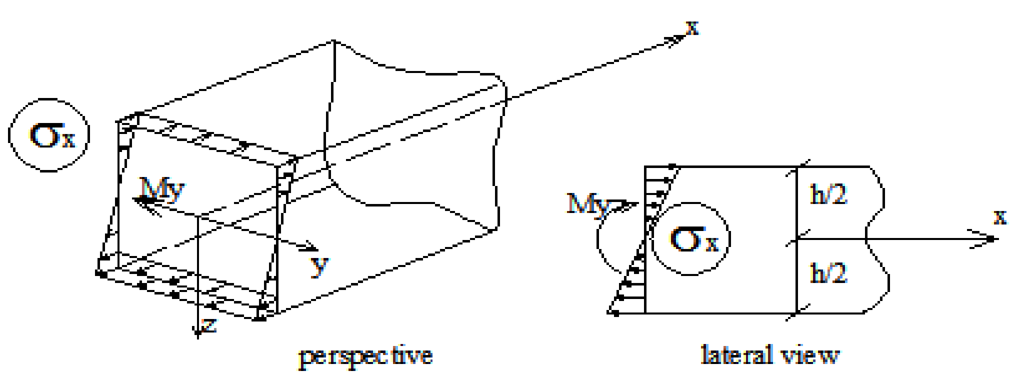


Fig.7.6

From Navier's relation we may define also ***the curvature of the neutral surface***: $\frac{1}{\rho} = \frac{M_y}{EI_y}$, which is in fact a deformation specific to pure bending, representing the relative rotation around the neutral axis of two cross sections situated to a unit distance one from another.

Claude-Louis Navier (1785-1836)

EI_y: the modulus of rigidity in bending, in [$daNcm^2$]

Navier's formula permit the calculation of the normal stress σ_x in any point of the cross section including the maximum value of σ_x (generally this is the most important value of σ_x in all design problems). The maximum normal stress σ_{xmax} correspond to the maximum coordinate z_{max}. As for the rectangular cross section in the upper and lower fiber the coordinate z are equal but of contrary signs, in these fibers σ_x will be maximum, one of tension (+) and the other compression (-):

$$\sigma_{xmax} = \frac{M_y}{I_y} z_{max} = \frac{M_y}{\frac{I_y}{z_{max}}}$$

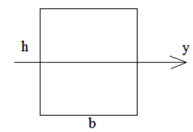
We may define *the strength modulus* (*section modulus*):

$$W_y = \frac{I_y}{z_{max}} \ [cm^3] \quad \text{and} \quad \sigma_{xmax} = \frac{M_y}{W_y}$$: another form of Navier's formula

For some cross sections:

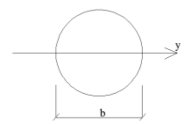
- rectangle:

$$W_y = \frac{I_y}{z_{max}} = \frac{bh^3}{12}\frac{2}{h} = \frac{bh^2}{6}$$

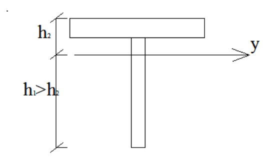


- circle:

$$W_y = \frac{\pi D^4}{64}\frac{2}{D} = \frac{\pi D^3}{32}$$



- I or U profile: $W_y = \frac{2I_y}{h}$ (given in tables)
- for a simple symmetrical C.S. :



$$W_{yb} = \frac{I_y}{h_1} > 0; \qquad W_{yt} = \frac{I_y}{h_2} < 0$$

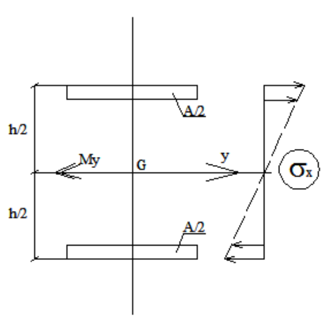
7.2.4 The rational sections for bent beams

If we observe once again Navier's formula: $\sigma_{xmax} = \frac{My}{Wy}$, we may admit that σ_x is inversely proportional to the section modulus W_y and implicit with the cross section height (in W_y the height h for the rectangle is at the second power) This means that the bent beams are recommended to have big heights (if the lateral stability is assured). Also the distribution of σ_x on the cross section height shows that the material is efficiently used if the cross section area is concentrated mostly in the extreme fibers.

We can introduce an ***index of efficiency***:

$$n = \frac{W_y}{A} = k \cdot h$$

where: k: is a coefficient that depends on the cross section shape



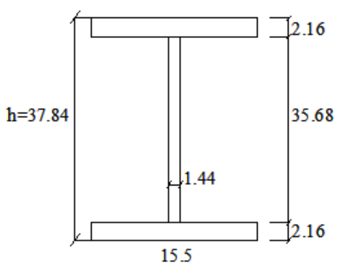
The greatest coefficient *n* corresponds to the most economical area, because the consumption of material is proportional to the cross section area. The optimum cross section of a bent bar is a hypothetic section made only from 2 flanges (rectangles). As the thickness of both flanges is very small, the distribution of σ_x is practically uniform.

$$I_y \cong 2\ \frac{A}{2}\left(\frac{h}{2}\right)^2 = \frac{Ah^2}{4} \rightarrow W_y \cong I_y\ \frac{2}{h} = \frac{Ah}{2}$$

and: $n = \frac{W_y}{A} = \frac{h}{2} = k\ h \ \rightarrow \mathbf{k = 0{,}5}$

In practice this solution is impossible (without a web to connect the two flanges), and the flanges must be connected by a web.

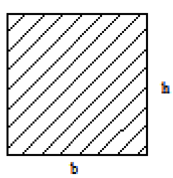
Ex: - for IPN 400:



$I_y = 29210\ cm^4$; $A = 118\ cm^4$; $W_y = 1460\ cm^3$

$n = \frac{1460}{118} = k \cdot 40 \rightarrow \mathbf{k = 0{,}31}$

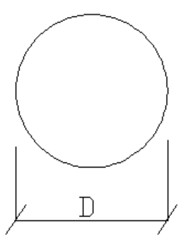
- for a rectangle :



$W_y = \frac{bh^2}{6}$; $A = bh$

$n = \frac{h}{6} = k \cdot h \rightarrow \mathbf{k = 0.17}$

- for a circle :



$W_y = \frac{\pi D^3}{32}$; $A = \frac{\pi D^2}{4}$

$n = \frac{D}{8} = k \cdot D \rightarrow \mathbf{k = 0{,}125}$

The most advantageous cross section for a bent beam is the I profile (it has the biggest coefficient k), because the area is distributed far off the neutral axis *Gy*.

The circular section is very disadvantageous for a bent element because the material is concentrated around the neutral axis, where σ_x is very small (a lot of material in the weakest subjected zone)

The efficiency of a bent section can be appreciated also by the lever arm of the stresses h_0 (Fig.7.7) defined as the distance between the resultants of the tensile stresses T and compressive stresses C.

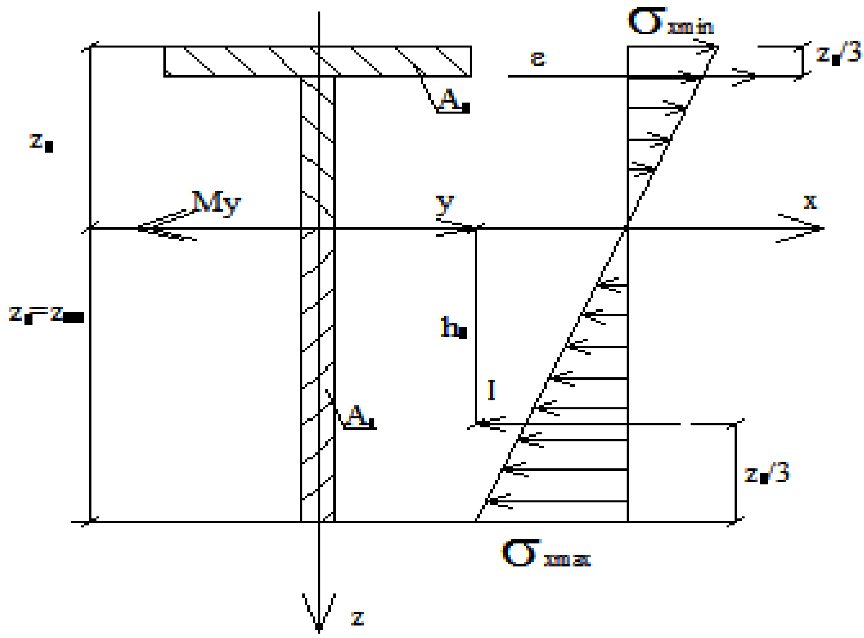


Fig.7.7

The two resultants are:

$$T = \int_{A_t} \sigma_x \, dA = \frac{M_y}{I_y} \int_{A_t} z \, dA = \frac{M_y}{I_y} S_{yt}$$

$$C = \int_{A_c} \sigma_x \, dA = \frac{M_y}{I_y} S_{yc}$$

As Gy is a central axis, $S_y = 0$ and $S_{yt} = - S_{yc}$ $\rightarrow$ $\mathbf{T = - C}$

These resultants form a couple which is exactly the bending moment from section:

$$M_y = T \cdot h_0 = C \cdot h_0$$

The level arm h_0, is:

$$h_0 = \frac{M_y}{T} = \frac{M_y}{C} = \frac{I_y}{S_{yt}} = \frac{I_y}{S_{yc}}$$

S_{yt} and S_{yc}: are the first moment of area (static moment) of the tensioned, respectively compressed area

- for the ideal section at bending :

$$S_{yt} = S_{yc} = \frac{A}{2}\frac{h}{2} = \frac{Ah}{4}; \qquad I_y = \frac{Ah^2}{4}$$

$$\Rightarrow h_0 = \frac{I_y}{S_y} = h$$

- for the double T cross section IPN 400:

$$S_{yt} = S_{yc} = 15{,}5 \times 2{,}16 \times 18{,}92 + 17{,}84 \times 1{,}44 \times 8{,}92 = 862{,}6 \text{ cm}^3$$

$$h_0 = \frac{29210}{862{,}6} = 33{,}86 \cong 0{,}85 \text{ h}$$

- for the rectangular C.S.:

$$S_{yt} = S_{yc} = b \times \frac{h}{2} \times \frac{h}{4} = \frac{bh^2}{8}$$

$$h_0 = \frac{bh^3}{12} \times \frac{8}{bh^2} = \frac{2}{3} h \cong 0{,}67 \text{ h}$$

We may conclude that if h_0 is closer to h, the cross section is rational for bending.

7.2.5 The main problems of design

In what concern the strength calculation (σ_x), the main aspects in designing the bent beams, are:

a) Verification

$$\sigma_{xmax} = \frac{M_{y\,max}^d}{W_y} \leq R$$

with: $M_y^d = n\,M_y^k$: is the design bending moment

M_y^k : is the characteristic bending moment

n: coefficient of loading (partial safety coefficient for load)

b) Dimensioning

$$W_{y\,nec} \geq \frac{M_{y\,max}^d}{R}$$

c) The bearing capacity at bending (the capable maximum bending moment)

$$M^d_{y\,max} \leq W_y \times R$$

7.3 STRAIGHT BENDING WITH SHEARING

Transverse loading applied to a beam produces normal stresses σ_x , but also shearing stresses τ_x in transverse cross sections. The normal stresses σ_x are produced by the bending moments M_y, while the shearing stresses τ_x are produced by the shear force V_z , when the force line is parallel to *Gz* axis (Fig.7.8) and passes through the shear center *C*. Otherwise, when the force line is parallel to *Gy* axis and passes through the shear center *C*, the normal stresses σ_x are produced by the bending moments M_z, while the shearing stresses τ_x are produced (Fig.7.8) by the shear force V_y. These stresses are represented in the figures bellow, with their positive convention. Now, this is a compound action, but it will be studied separately from each stress.

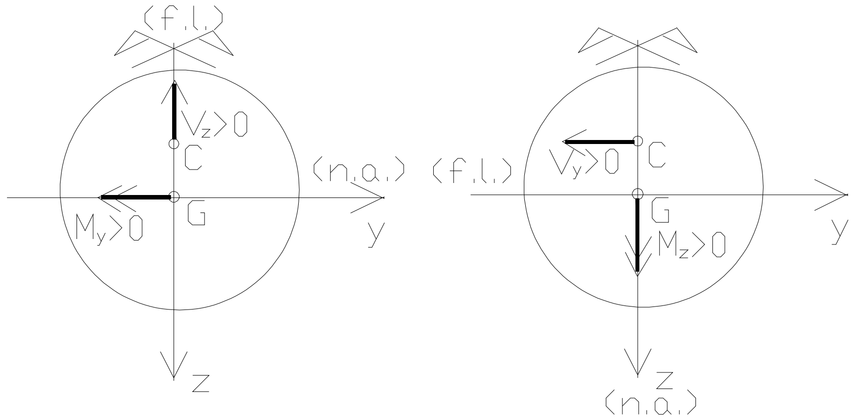


Fig.7.8

The effect of the bending moment M_y was presented in the previous chapter, *pure bending*, obtaining finally Navier's formula for the normal stress σ_x. The shear force V_z will produce a tangential (shear) stress τ_x, presented below.

7.3.1 Straight shearing (sliding)

7.3.1.1 Geometrical aspect

To explain the effect of shear force we consider a model (Fig.7.9) of a bar loaded in such manner so that a part of the bar is subjected to pure bending and the rest at bending with shearing (example b.) from the first paragraph).

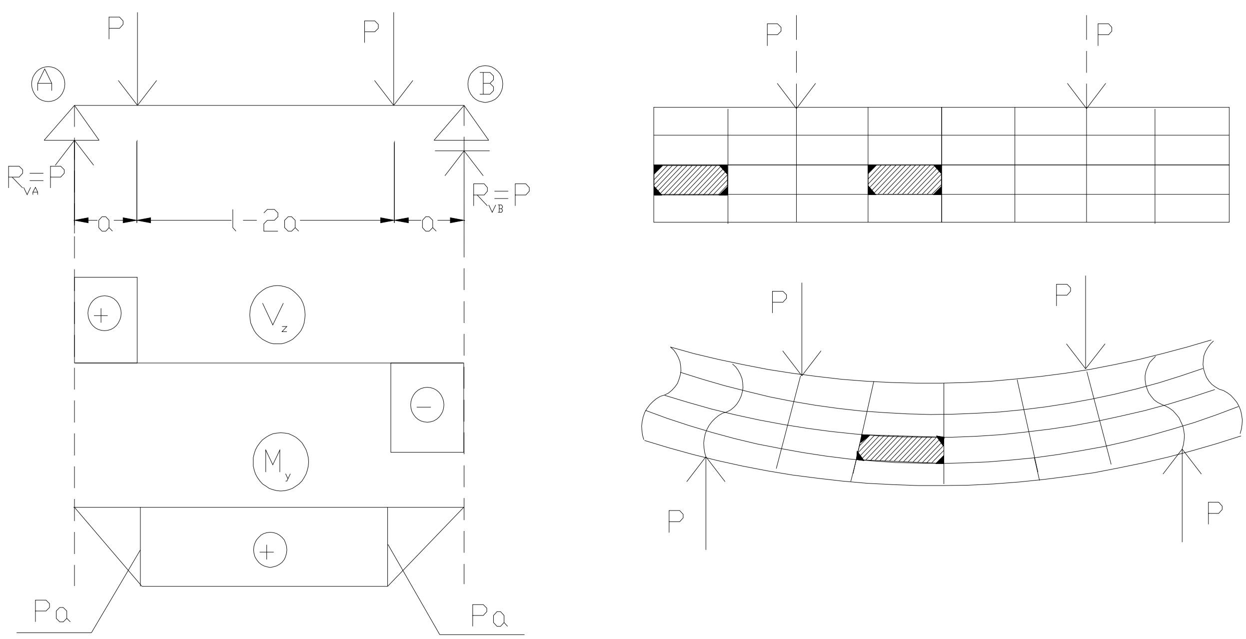
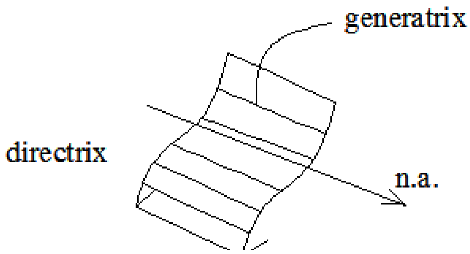


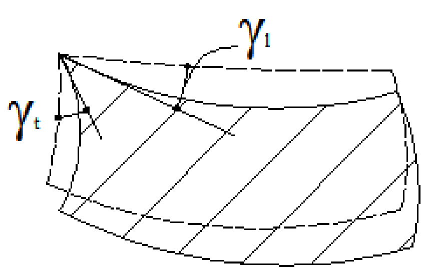
Fig.7.9

Considering that the network from the lateral surface of the model is valid inside the model, we observe:

1. a supplementary increasing of the curvature of the longitudinal lines, but with a small quantity (approximately 5% from the curvature produced by the bending moment M_y)
2. in the central zone of length *"l-2a"* the cross sections remain plane and perpendicular to the longitudinal curved fibers.
3. in the marginal zones of length *"a"* subjected to bending with shearing, the cross sections are distorted, becoming cylindrical surfaces with the generators: straight lines and the directories: curves with a *S* shape, which remain perpendicular only to the extreme fibers.



4. analyzing a rectangle from network from this zone of length *"a"*, we remark that the distortion is introduced by angular deformations (specific sidings) γ_{xz} in plan parallel to *xGz* plan, the initial straight angles being modified with:

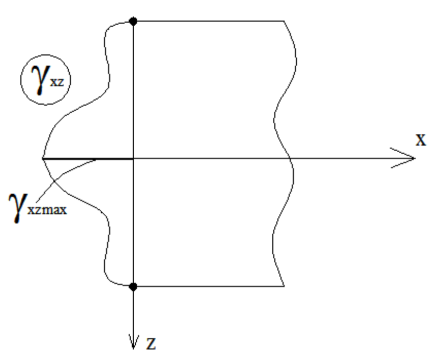


$$\gamma_{xz} = \gamma_t + \gamma_l$$

γ_t : is produced by the relative transversal sliding of the cross section

γ_l : is produced by the relative longitudinal sliding of the longitudinal fibers parallel to the neutral surface (strip)

We may observe from model that the specific sliding γ_{xz} isn't constant on the cross section height, being maximum in the neutral strip and zero in the extreme fibers, where the straight angle was maintained.



But, on the height of the cross section the specific sliding γ_{xz} has an unknown distribution, as well as on the cross section width. But, as to any level *z*, the tangent to the directories has the same inclination the specific sliding is constant on the cross section width.

7.3.1.2 Physical aspect

A material with a linear-elastic behaviour is considered, where Hooke's law is valid: $\tau_{xz} = G \times \gamma_{xz}$

So, the shear stresses τ_{xz} have the same distribution as γ_{xz}, being constant on the cross section width at any level z, but with an unknown variation on the cross section height. Representing the constant distribution of τ_{xz} in the cross section plan, we may admit in accordance to the duality law that shear stresses τ_{zx} ($\tau_{xz} = \tau_{zx}$) will also exist, in longitudinal plans which are parallel to the neutral strip (Fig.7.10). Longitudinal shearing stresses must exist in any member subjected to transverse loading.

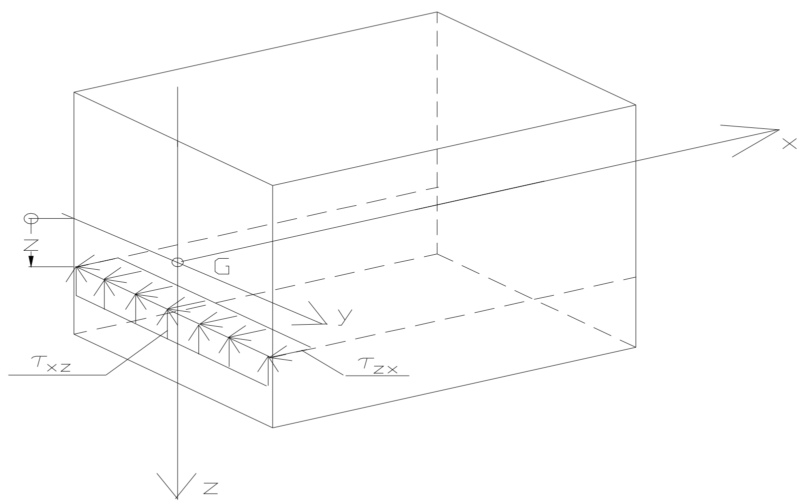


Fig.7.10

These two ***complementary stresses*** τ_{xz} and τ_{zx} correspond to 2 types of sliding: transversal sliding and longitudinal sliding**.** Shorter we'll say that τ_{xz} produces ***shearing*** and τ_{zx} produces ***sliding*.**

7.3.1.3 Static aspect

The shear force V_z from static calculation is: $V_z^{st} = P$

From strength calculation, V_z is: $V_z^{res} = \int_A \tau_{xz} dA$, unknown because the distribution of τ_{xz} is unknown on the cross section height.

That's why we have to search another cross section on which the shear stress distribution is known.

7.3.1.4 Formula of Juravski

We'll consider a bar loaded by a system of forces acting perpendicularly to the axis and comprised in the longitudinal symmetry plan of the bar (Fig.7.11).

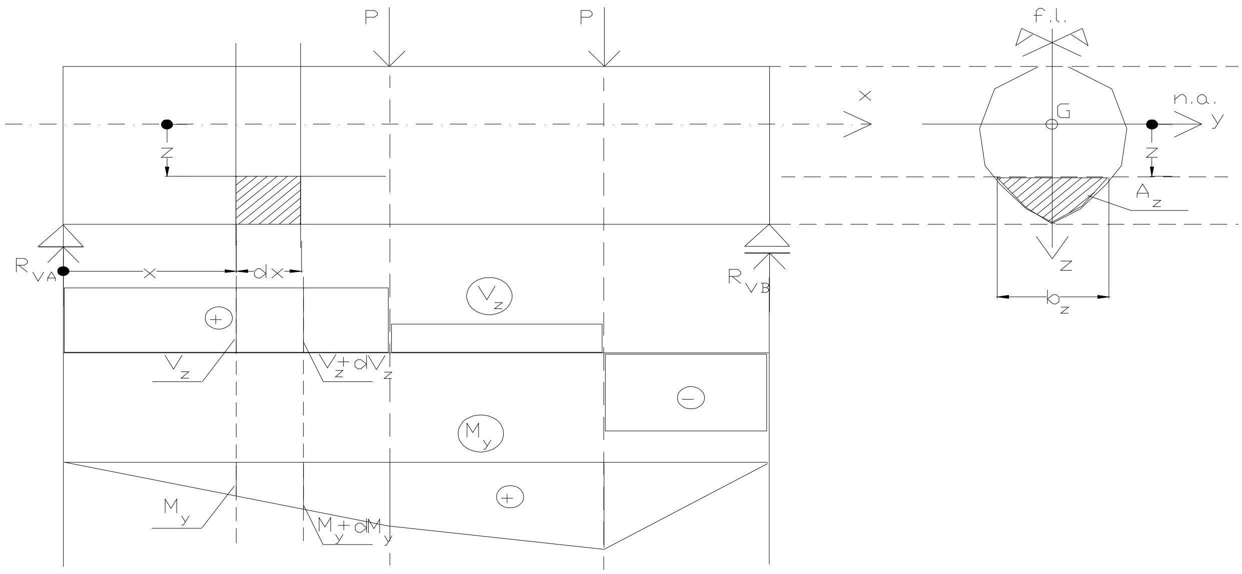


Fig.7.11

We may consider a constant distribution of τ_{xz} on a differential element of length dx, situated to a level z from the neutral strip (Fig.7.12).

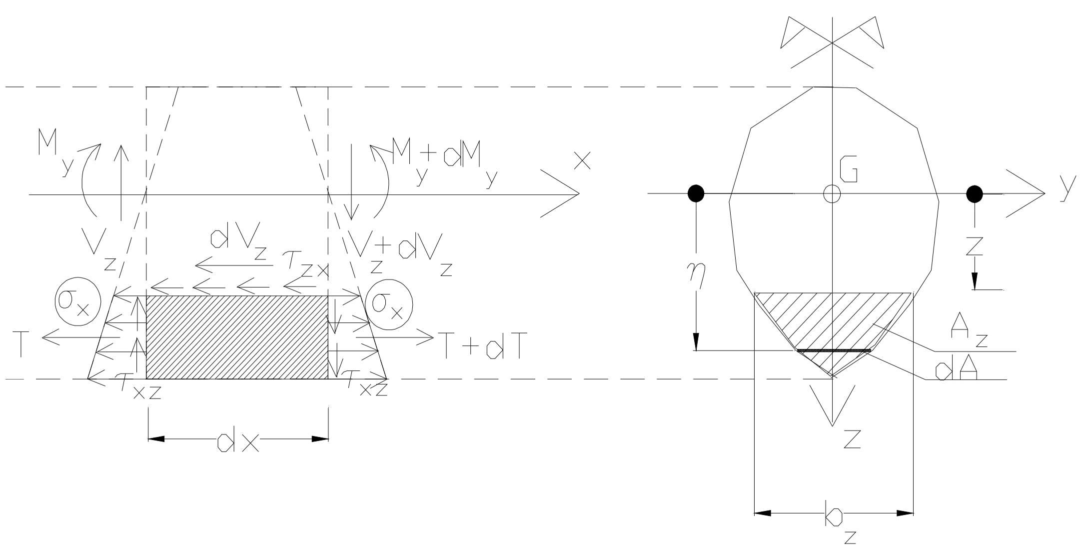


Fig.7.12

According to the method of sections, the effect of the removed part is introduced by the stresses acting on the differential element. So, on the transversal sections we dispose the normal stresses σ_x and the shear stresses τ_{xz}, while on the longitudinal section at level z we dispose the sliding stresses τ_{zx}.

To write equations of static equilibrium for this differential element, first we have to write the resultants of the stresses acting on the differential element.

At a level η the normal stress σ_x is:

$$\sigma_x = \frac{M_y}{I_y} \cdot \eta$$

and the resultant:

$$T = \int_{A_z} \sigma_x dA = \frac{M_y}{I_y} \int_{A_z} \eta \cdot dA = \frac{M_y}{I_y} S_y(z)$$

where: $S_y(z)$: is the static moment, written about the neutral axis Gy, of the section A_z situated to a level z, called **calculus level**.

The differential resultant dT is (if the cross section of the bar is constant):

$$dT = d\left(\frac{M_y}{I_y} S_y(z)\right) = \frac{S_y(z)}{I_y} dM_y = \frac{S_y(z)}{I_y} V_z dx$$

where: $dM_y = V_z\, dx$, from the second differential relation between stresses.

The resultant of the sliding stress τ_{zx} is:

$$dL_z = \tau_{zx} \cdot b_z \cdot dx$$

We write an equation of static equilibrium along the bar axis:

$\Sigma x = 0$: $-T + T + dT - dL_z = 0 \Rightarrow dT = dL_z$

Replacing:

$$\frac{S_y(z)}{I_y} V_z\, dx = \tau_{zx}\, b_z\, dx, \text{ but } \tau_{zx} = \tau_{xz}$$

$$\boxed{\tau_{xz} = \frac{V_z S_y(z)}{b_z I_y}}$$ **Juravski's formula**

Dmitrii Ivanovich Juravskii (1821-1891)

The formula shows that the shear stresses τ_{xz} are proportional to the shear force V_z and they have the same orientation as V_z, in cross section. The distribution of τ_{xz} on the cross section height is given by the variation of the ratio $\frac{S_y(z)}{b_z}$. As in the extreme fibers $S_y(z) = 0$, the shear stress τ_{xz} is also null.

7.3.1.5 Juravski's formula for the narrow rectangular section

A rectangular section is narrow, from shear point of view, if: $\frac{h}{b} \geq 2$

At a level z, the static moment (Fig.7.13) is:

$$S_y(z) = b(\frac{h}{2} - z)(z + \frac{h}{4} - \frac{z}{2}) = b(\frac{h}{2} - z)(\frac{z}{2} + \frac{h}{4}) = \frac{b}{2}(\frac{h}{2} - z)(\frac{h}{2} + z) = \frac{b}{2}(\frac{h^2}{4} - z^2)$$

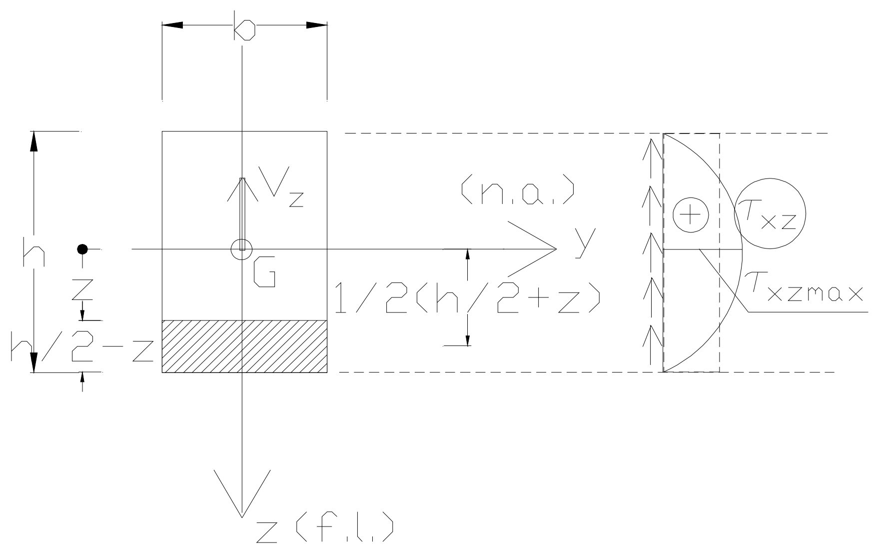


Fig.7.13

The shear stress:

$$\tau_{xz} = \frac{V_z S_y(z)}{b_z I_y} = \frac{V_z \frac{b}{2}(\frac{h^2}{4} - z^2)}{b\frac{bh^3}{12}} = \frac{6V_z}{bh^3}(\frac{h^2}{4} - z^2)$$

The above expression shows that τ_{xz} has a parabolic variation on the cross section height.

For $z = \pm\frac{h}{2}$ => $\tau_{xz} = 0$

For $z = 0$ => $\tau_{xz\,max} = \frac{6V_z}{bh^3}\frac{h^2}{4} = \frac{3}{2}\frac{V_z}{bh}$

$$\boxed{\tau_{xz\,max} = 1{,}5\,\frac{V_z}{A}}$$

For the rectangular cross section the maximum shear stress from shearing with bending is with 50% bigger that the medium stress $\frac{V_z}{A}$ from pure shearing.

7.3.1.6 Juravski's formula for a double T cross section, made from narrow rectangles

From previous paragraph we may remark that for a narrow rectangle the shear stresses are always orientated along the longest side of the rectangle. This observation is also valid for the cross sections made from narrow rectangles (the rolled, laminated profiles), the shear stresses τ_x being orientated along the biggest side of each rectangle, whatever is the relative position between this side and the force line. In accordance to this observation, in a double T cross section τ_{xz} will exist in web (the longest side of the web is parallel to Gz axis), while in flanges a shear stress τ_{xy} will appear (the flange is parallel to Gy axis)

Let's consider a double symmetrical I section (Fig.7.14), subjected to straight shearing by a positive shear force $V_z > 0$.

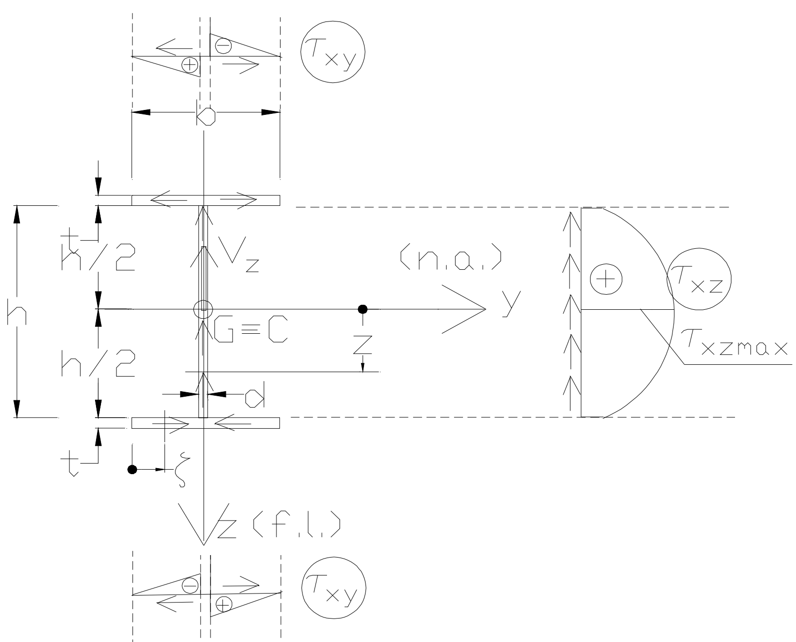


Fig.7.14

$$S_y(z) = b\,t\left(\frac{h}{2} + \frac{t}{2}\right) + d\left(\frac{h}{2} - z\right)\left(\frac{h}{4} - \frac{z}{2} + z\right) = \frac{bt}{2}(h + t) + \frac{d}{2}\left(\frac{h^2}{4} - z^2\right)$$

$$\tau_{xz}(z) = \frac{V_z}{2dI_y}[bt(h+t) + d(\frac{h^2}{4} - z^2)]$$

The expression of τ_{xz} is mathematically a parabola of second degree, so in web the shear stress τ_{xz} parallel to *z* will have a parabolic variation. The maximum value will correspond to z = 0 (in the neutral axis):

$$\tau_{xz}(z) = \frac{T_z}{2dI_y}[bt(h+t) + \frac{d\,h^2}{4}]$$

and, for $z = \pm\frac{h}{2}$:

$$\tau_{xz}(z) = \frac{T_z}{2dI_y}\,bt(h+t)\text{ , a smaller value}$$

So, the shear stress τ_{xz} is drawn on the cross section web (parallel to *z* axis), it has a parabolic variation on the web height and with a maximum value in the neutral axis. As the thickness *t* of the rectangle is very small we admit that the shear stresses τ_x are uniformly distributed on thickness (it is constant), and their resultant $\tau_x \times t$ is called **shearing flow**. To explain the shearing flow in a thin-walled section, we start from τ from web (τ_{xz}) which has the same direction as V_z and then a hydrodynamic analogy is made considering that the section is a thin-walled tube and a liquid must flow through it.

In flange, at the level ξ:

$$S_y(\xi) = \xi\, t\,(\frac{h}{2} + \frac{t}{2})$$

$$\tau_{xy}(\xi) = \frac{V_z}{2I_y}\,\xi(h+t) \;\; => \tau_{xy} \text{ has a linear variation}$$

For $\xi = 0 => \tau_{xy} = 0$

For $\xi = \frac{b-d}{2} => \tau_{xy} = \frac{V_z(b-d)(h+t)}{4I_y}$

7.3.1.7. Juravski's formula for a tubular cross section, made from narrow rectangles

We consider a double symmetrical tubular (caisson) cross section (Fig.7.15).

The static moments:

$$S_y(z) = bt(\frac{h}{2} - \frac{t}{2}) + 2(\frac{h}{2} - t - z)\, d\, (\frac{h}{4} - \frac{t}{2} - \frac{z}{2} + z) = \frac{bt}{2}(h - t) + [(\frac{h}{2} - t)^2 - z^2]d$$

$b_z = 2d$

$$S_y(\xi) = \frac{t\,\xi}{2}(h - t)$$

$b_z(\xi) = t$

The shear stress at the level *z*:

$$\tau_{xz}(z) = \frac{V_z}{2dI_y}\left\{\frac{bt}{2}(h - t) + [\left(\frac{h}{2} - t\right)^2 - z^2]d\right\}$$

The shear stress at the level ξ:

$$\tau_{xy}(\xi) = \frac{V_z}{2I_y}\xi(h-t)$$

In the symmetry axis *Gz*, when the shear force V_z acts along it, $\tau_{xy} = 0$

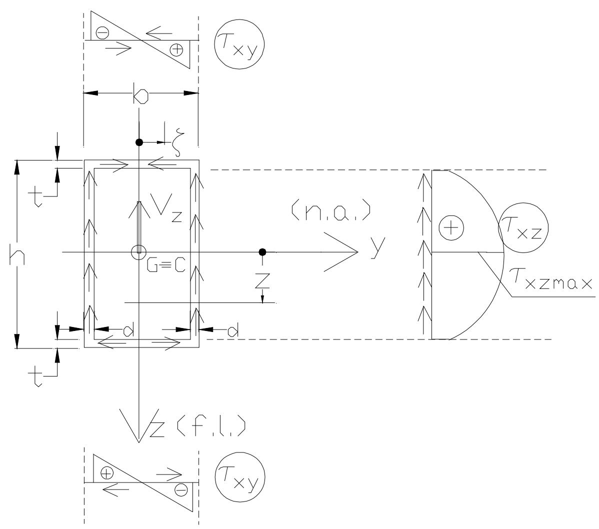
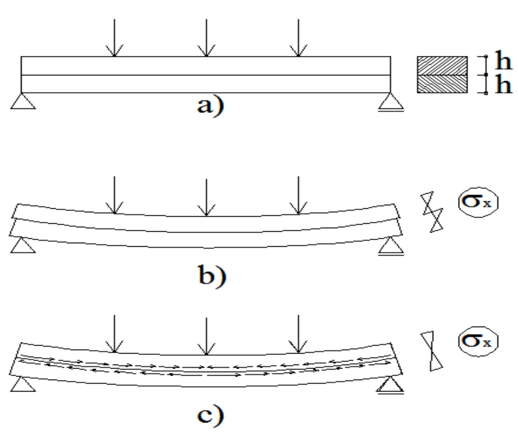


Fig.7.15

7.3.2 Longitudinal sliding

7.3.2.1 The longitudinal force of sliding

In the previous paragraph, we have seen that in accordance to duality law, tangential (sliding) stress τ_{zx} will also exist in plans that are parallel to the neutral surface, called shorter **sliding stresses** τ_{zx}. On a length of beam, their resultant is called **force of sliding.**



We can explain very simple the existence of these sliding forces, considering a composed beam made from 2 superposed elements (a).

In the first situation we admit that the 2 elements aren't connected (b). Each element is deformed separately, generating separately the typical linear variation of normal stress σ_x over its own depth. The contact surfaces will slide one with respect to the other.

Now, considering the elements are connected between them, both elements respond as a unit (c). Bending stresses will now vary linearly over the whole depth. The relative sliding of the contact surfaces is prevented, along these surfaces appearing sliding forces.

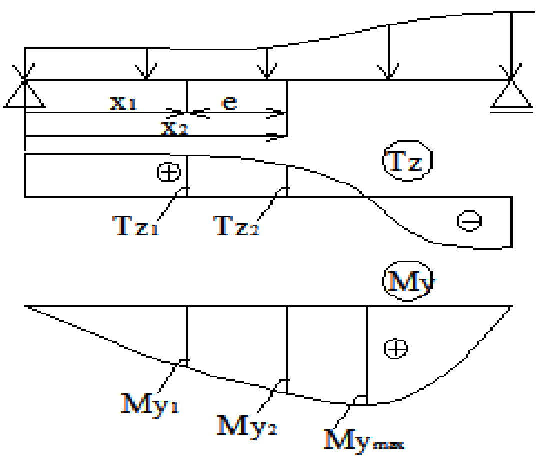
To evaluate ***the sliding force***, we start from the differential sliding force dL_z used in the demonstration of Juravski's formula:

$dL_z = \tau_{zx}\, b_z\, dx$, at a calculus level z

but: $\tau_{zx} = \frac{V_z S_y(z)}{b_z I_y}$

and: $dL_z = \frac{V_z S_y(z)}{I_y}\, dx$

On a finite interval, between x_2 and x_1, from integration, the sliding force is:



$$L_z = \int_{x_1}^{x_2} dL_z = \int_{x_1}^{x_2} \frac{V_z S_y(z)}{I_y}\, dx$$

If the cross section is constant on the length $e = x_2 - x_1$, the ratio $\frac{S_y(z)}{I_y}$ = const.

$$L_z = \frac{S_y(z)}{I_y} \int_{x_1}^{x_2} V_z\, dx$$

With: $dM_y = V_z \cdot dx$ =>

$$L_z = \frac{S_y(z)}{I_y} \int_{x_1}^{x_2} dM_y = \frac{S_y(z)}{I_y} (M_{y2} - M_{y1})$$

If the distance: $e = x_2 - x_1$ is small we may consider a constant distribution of the shear force V_z on this distance, and:

$$L_z = \frac{S_y(z)}{I_y} V_z \int_{x_1}^{x_2} dx = \frac{S_y(z)}{I_y} V_z (x_2 - x_1) \Rightarrow$$

$$\boxed{\mathbf{L_z = \frac{S_y(z)\, V_z}{I_y}\, e}}$$

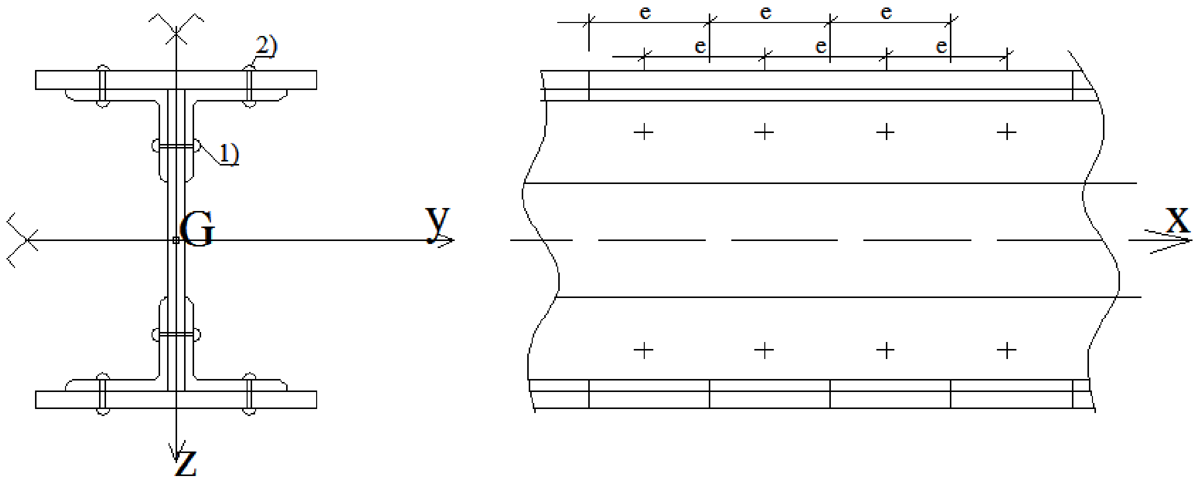
In this formula:

$S_y(z)$ is the static moment of the area that slides longitudinally, admitting that we neglect the connection elements (rivets, bolts, welding).

e: the longitudinal distance between these elements (rivets)

7.3.2.2 The elements of connection

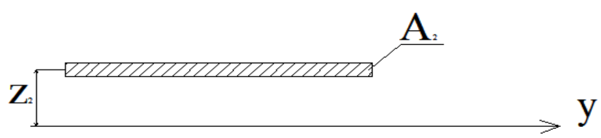
We discuss the case of a composed I section, connected with rivets through 4 angles with equal legs. The angles are connected to web by **grove rivets (1)**



and to flanges by **head rivets (2).** From constructive reasons, the diameters of both types of rivets are taken identically, but the head rivets are placed in sections situated to a semi distance (e/2) from the sections with groove rivets (to avoid a greater diminishing of the cross section due to the rivets holes)

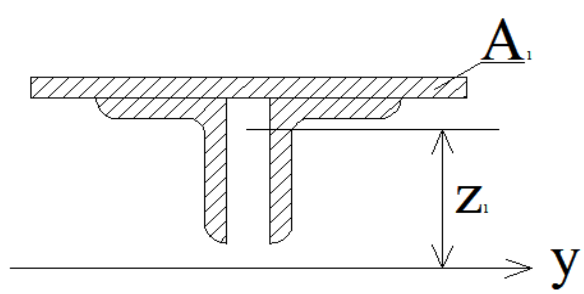
In the formula of L_z, the static moment $S_y(z)$ is taken :

- for head rivets (2) :



$S_y(z_2) = A_2\ z_2$

- for grove rivets (1):



$S_y(z_1) = A_1\ z_1 > S_y(z_2)$

As S_y for the grove rivets (1) is bigger: $S_y(z_1) > S_y(z_2)$ the sliding force is also bigger $L_{z1} > L_{z2}$, so it is sufficient if we compute only these rivets (1):

$$L_{z\,1} = \frac{S_y(z1)\ V_z}{I_y}\, e$$

This sliding force, which corresponds to one rivet, **must not exceed** the minimum stress (force) which can be transmitted by one rivet, representing the minimum between the stress corresponding to the rivet shearing, respectively the stress corresponding to the local pressure:

$N_{min} = \min (N_f, N_p)$

$N_f = n_f \frac{\pi d^2}{4} 0.8R$

$N_p = d (\boldsymbol{\Sigma} t)_{min} 2R$

So, the condition is:

$$L_{z1} \leq N_{min} \quad \Rightarrow \quad e \leq \frac{N_{min} I_y}{V_z S_y(z_1)}$$

where: I_y and $S_y(z_1)$ are taken with their gross values (without diminishing)
Supplementary, there are constructive measures given in standards, which impose:

$e \leq 8d$

$e \leq 12t$

where : d: the rivet (bolt) diameter
t: the minimum thickness of the elements connected with rivets (bolts)

7.4. THE CALCULATION OF DEFLECTIONS DUE TO UNIAXIAL BENDING

7.4.1. Basis of design

Let's consider a simple supported beam subjected to uniaxial (straight) bending (fig.7.16).

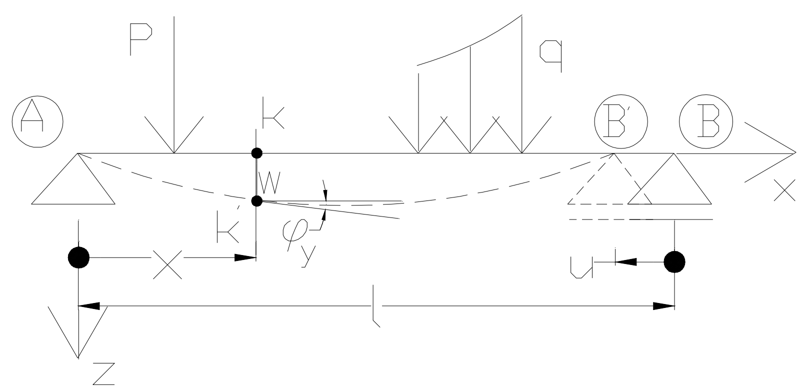


Fig.7.16

The beam is represented in a deformed shape, the main deformations being inscribed on figure.

Admitting the hypothesis of the small deformations, the calculation of these deformations is made on the undeformed shape of the bar (calculus of first order) and we shall admit that we have only vertical **displacements *w***, called **deflections** (the longitudinal displacement *u* of the mobile support is neglected and horizontal component of the displacement u_k is also neglected in comparison with the deflection w_k). Also, in time of deformation, the cross sections are rotating around the neutral axis Gy, producing the rotation φ_y, which is equal to the angle between the horizontal and the tangent to the deformed shape of the beam.

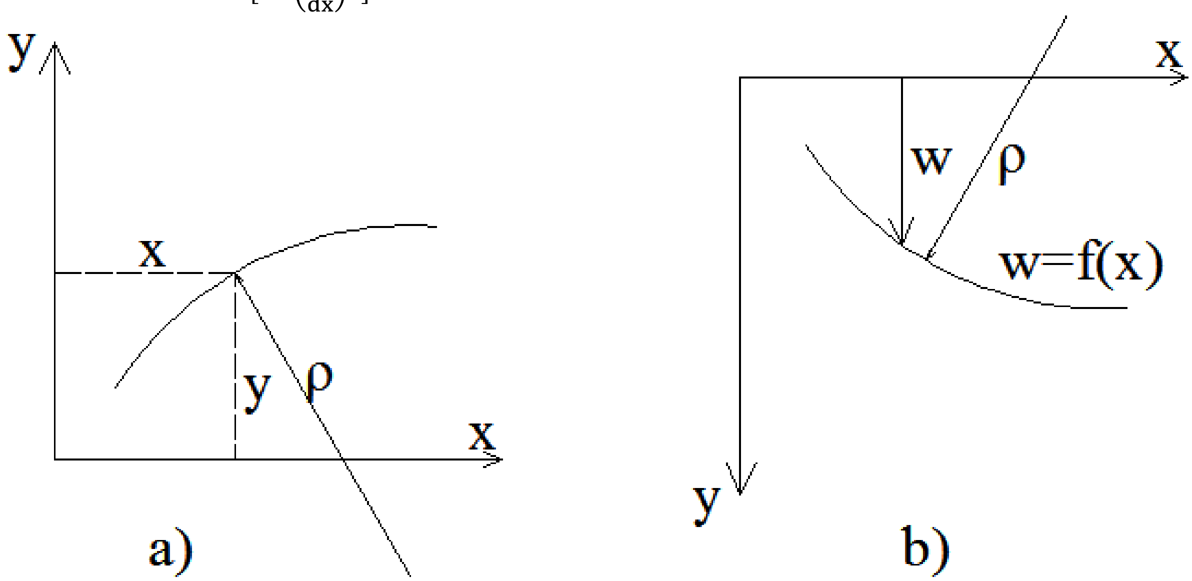
The displacement (deflections and rotations) of the cross section can be calculated by two main methods:

- methods that use the differential equation of the deformed axis (fiber)
- energetically methods (studied later)

7.4.2 The analytical calculation of the displacements integrating the differential equation of the deformed axis (Direct integration of the equation of the elastic curve)

From analytical geometry, the expression of the curvature of a plane curve (a) is:

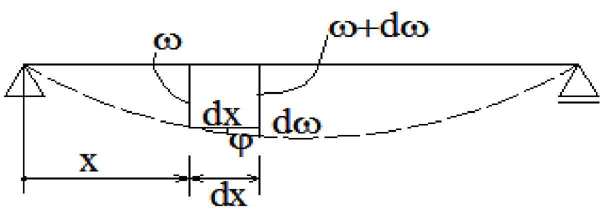
$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{[1+\left(\frac{dy}{dx}\right)^2]^{3/2}}$$



For a curved bar (b) the curvature can be written:

$$\frac{1}{\rho} = \frac{\frac{d^2w}{dx^2}}{[1+\left(\frac{dw}{dx}\right)^2]^{3/2}}$$

In practice the deformed fiber (axis) has a very small curvature and the first derivative of the deflection w can be assimilated to the rotation φ:



$$\frac{dw}{dx} = w' = tg\varphi \cong \varphi$$

With this, the curvature in the above relation: $\frac{1}{\rho} = \frac{\frac{d^2w}{dx^2}}{(1+\varphi^2)^{3/2}}$

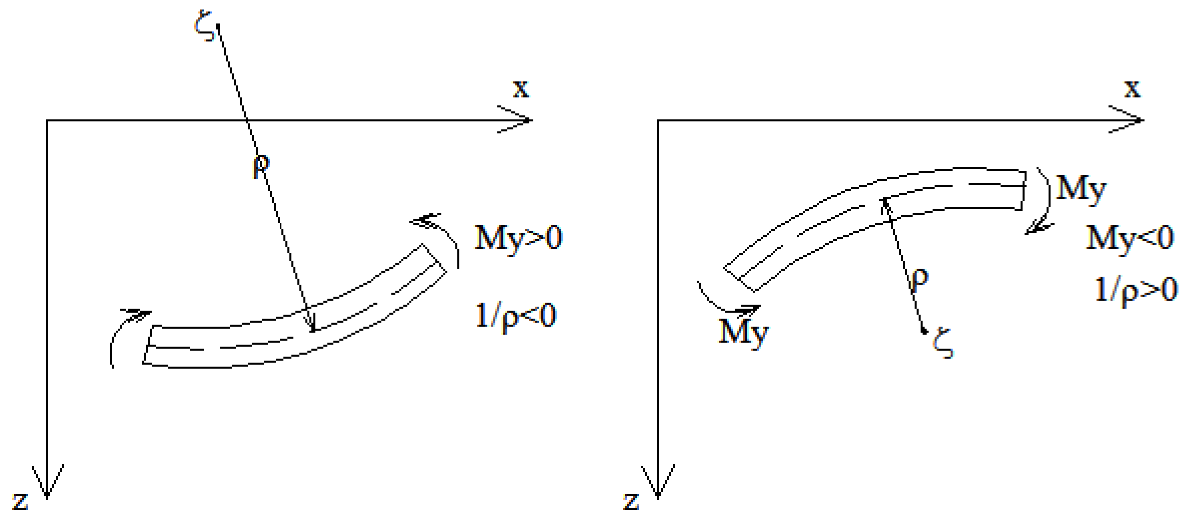
But, the rotation φ, expressed in radian, has always a very small value, so the square of φ in this relation can be neglected. With this simplification, the curvature is:

$$\frac{1}{\rho} = \frac{d^2w}{dx^2} = w''$$

But, from pure bending we know that the relationship between bending moment and curvature remains valid for general transverse loadings:

$$\frac{1}{\rho} = \frac{M_y}{EI_y}$$

Between the curvature $\frac{1}{\rho}$ and the bending moment M_y, a sign convention is considered: the curvature is positive if the curvature center C_ρ is situated towards positive z axis:



In conclusion: the curvature $\frac{1}{\rho}$ and the bending moment will always have different signs.

From this observation we may write now correct the relationship between bending moment and curvature, obtained in paragraph 7.2.3:

$$\frac{1}{\rho} = -\frac{My}{EIy}$$

Identifying these two expressions of the curvature we write finally the differential **equation of the deformed axis (equation of the elastic curve):**

$$\frac{d^2w}{dx^2} = w'' = -\frac{My}{EIy}$$ where: EI_y is the modulus of rigidity for bending

Integrating once, the **rotation φ** is obtained:

$$\frac{dw}{dx} = \varphi = -\int \frac{M_y}{EI_y} dx + C_1$$

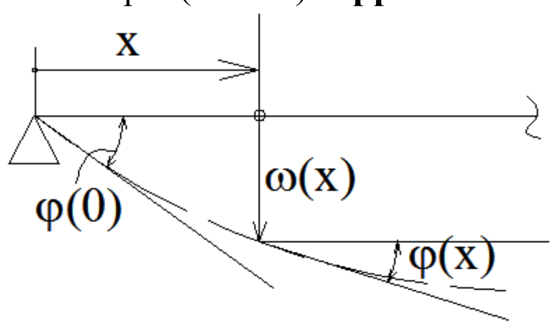
Integrating once again, the **deflection *w*** is obtained:

$$w = -\int dx \int \frac{M_y}{EI_y} dx + C_1 x + C_2$$

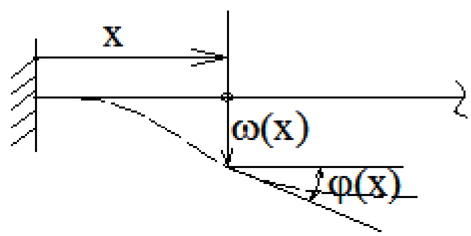
The constants of integration C_1 and C_2 can be calculated from ***boundary conditions*** written in supports, or from ***continuity conditions***.

Boundary conditions:

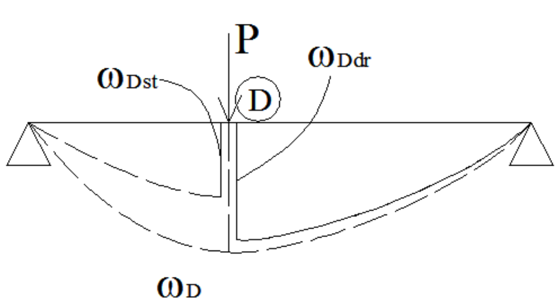
- for simple (mobile) **support** or for **hinge**: $\boxed{\mathbf{w(0) = 0}}$, $\varphi(0) \neq 0$



- for a **built-in** (fixed) **support:** $\boxed{w(0) = 0}$, $\boxed{\varphi(0) = 0}$



The **continuity conditions** take account the fact that the deformed shape of the bar must remain a continuous function and with continuous derivatives.

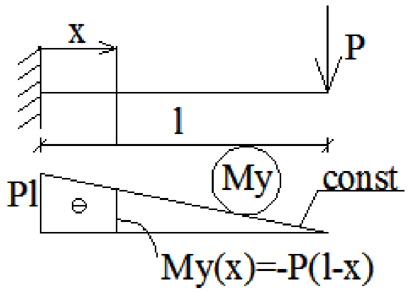


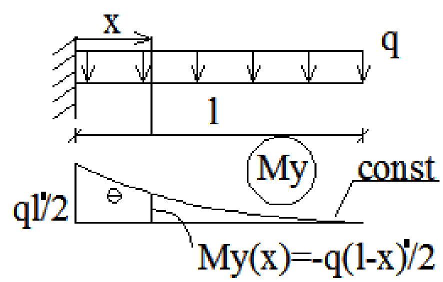
- the continuity conditions:

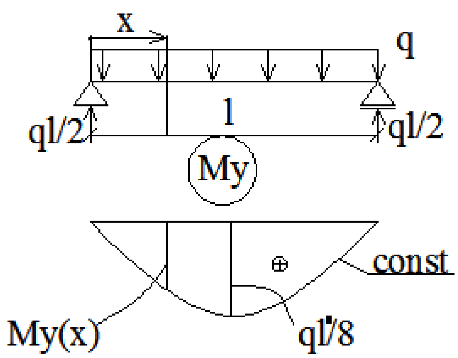
$w_{Dst} = w_{Ddr}$ and $\varphi_{Dst} = \varphi_{Ddr}$

To integrate the differential equation $\frac{d^2w}{dx^2} = -\frac{My}{EIy}$, the function $M_y(x)$ must be continuous and with continuous derivatives.

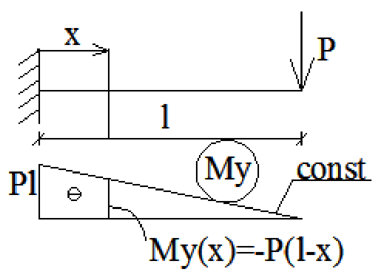
Examples:







For the cantilever loaded by the concentrate force P acting in the free end:



$M_y(x) = -P(l-x)$ which is introduced in the equation of the elastic curve:

$$\frac{\mathrm{d}^2 \mathrm{w}}{\mathrm{dx}^2} = -\frac{\mathrm{My}}{\mathrm{EIy}}$$

$$\frac{\mathrm{dw}}{\mathrm{dx}} = \varphi = \int \frac{P(l-x)}{EI_y} dx = \frac{Pl}{EI_y} x - \frac{P}{EI_y} \frac{x^2}{2} + c_1$$

$$\mathrm{w} = -\int dx \int \frac{M_y}{EI_y} dx = \frac{Pl}{EI_y} \frac{x^2}{2} - \frac{P}{EI_y} \frac{x^3}{6} + c_1 x + c_2$$

Writing the boundary conditions in the built-in support:

$w(0) = 0$ and $\varphi(0) = 0 \ \rightarrow c_1 = c_2 = 0$

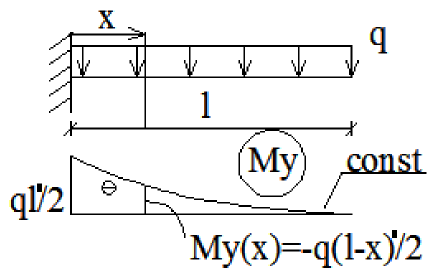
The analytical expressions of the deflection w(x) and rotation φ(x) are:

$\mathrm{w(x)} = \frac{Pl}{EI_y} \frac{x^2}{2} - \frac{P}{EI_y} \frac{x^3}{6}$ and $\varphi(\mathrm{x}) = \frac{Pl}{EI_y} x - \frac{P}{EI_y} \frac{x^2}{2}$

Their maximum values are in the free end of the cantilever, so for x=l:

$w_{max} = \frac{Pl^3}{3EI_y}$ and $\varphi_{max} = \frac{Pl^2}{2EI_y}$

For the cantilever loaded by the uniformly distributed force q on the entire length:



$M_y(x) = -q\frac{(l-x)^2}{2}$ which is introduced in the equation of the elastic curve:

$$\frac{d^2w}{dx^2} = -\frac{My}{EIy}$$

$$\frac{dw}{dx} = \varphi = \int q\frac{(l-x)^2}{2}dx = \frac{ql^2}{2EI_y}x - \frac{ql}{EI_y}\frac{x^2}{2} + \frac{q}{2EI_y}\frac{x^3}{3} + c_1$$

$$w = -\int dx\int\frac{M_y}{EI_y}dx \;= \frac{ql^2}{2EI_y}\frac{x^2}{2} - \frac{ql}{EI_y}\frac{x^3}{6} + \frac{q}{2EI_y}\frac{x^4}{12} + c_1x + c_2$$

Writing the boundary conditions in the built-in support:

$w(0) = 0$ and $\varphi(0) = 0 \;\rightarrow c_1 = c_2 = 0$

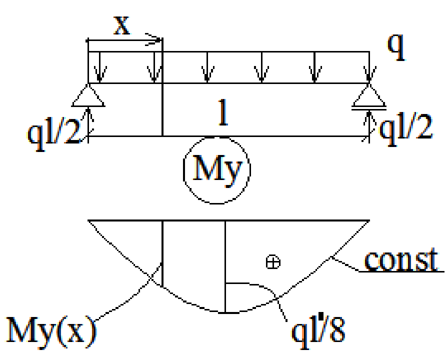
The analytical expressions of the deflection w(x) and rotation φ(x) are:

$w(x) = \frac{ql^2}{2EI_y}\frac{x^2}{2} - \frac{ql}{EI_y}\frac{x^3}{6} + \frac{q}{2EI_y}\frac{x^4}{12}$ and $\varphi(x) = \frac{ql^2}{2EI_y}x - \frac{ql}{EI_y}\frac{x^2}{2} + \frac{q}{2EI_y}\frac{x^3}{3}$

Their maximum values are in the free end of the cantilever, so for x=l:

$w_{max} = \frac{ql^4}{8EI_y}$ and $\varphi_{max} = \frac{ql^3}{6EI_y}$

For the simple supported beam loaded by the uniformly distributed force q on the entire length:



$M_y(x) = \frac{ql}{2}x - q\frac{x^2}{2}$ which is introduced in the equation of the elastic curve:

$$\frac{d^2w}{dx^2} = -\frac{My}{EIy}$$

$$\frac{dw}{dx} = \varphi = \int(-\frac{ql}{2}x + q\frac{x^2}{2})dx = -\frac{ql}{2EI_y}\frac{x^2}{2} + \frac{q}{EI_y}\frac{x^3}{6} + c_1$$

w = $- \int dx \int \frac{M_y}{EI_y} dx \; = -\frac{ql}{2EI_y}\frac{x^3}{6} + \frac{q}{EI_y}\frac{x^4}{24} + c_1 x + c_2$

Writing the boundary conditions in the simple support and in hinge:

$w(0) = 0$ and $w(l) = 0 \; \rightarrow c_2 = 0$ and $c_1 = \frac{ql^3}{24EI_y}$

The analytical expressions of the deflection w(x) and rotation φ(x) are:

$w(x) = -\frac{ql}{2EI_y}\frac{x^3}{6} + \frac{q}{EI_y}\frac{x^4}{24} + \frac{ql^3}{24EI_y}x$ and $\varphi(x) = -\frac{ql}{2EI_y}\frac{x^2}{2} + \frac{q}{EI_y}\frac{x^3}{6} + \frac{ql^3}{24EI_y}$

The maximum value of the deflection is in the middle span of the beam, so for $x=\frac{l}{2}$:

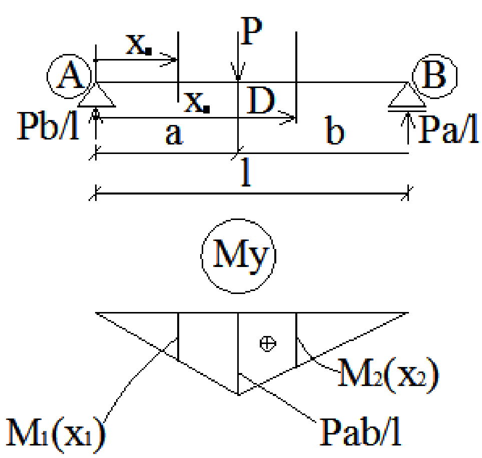
$w_{max} = \frac{5}{384}\frac{ql^4}{EI_y}$ and the corresponding rotation $\varphi\left(\frac{l}{2}\right) = 0$

The maximum value of the deflection is in the beam supports, so for x=0 or x=l:

$$\varphi_{max} = \pm\frac{ql^3}{24EI_y}$$

If M_y diagram isn't continuous, the integration must be done on each interval of continuity.

Example:



$$M_1(x_1) = \frac{Pb}{l}x_1$$

$$M_2(x_2) = \frac{Pb}{l}x_2 - P(x_2 - a)$$

7.4.3 Conjugate beam method

The method uses the formal analogy between the differential equation of the deformed axis:

$$\frac{d^2w}{dx^2} = \frac{d\varphi}{dx} = -\frac{M}{EI}$$

and the differential relations between stresses and loads (from statics):

$$\frac{d^2M}{dx^2} = \frac{dV}{dx} = -p$$

We can formally compare:

The displacement w with the moment $\overline{M}$

The rotation (slope) φ with the shear force $\overline{V}$

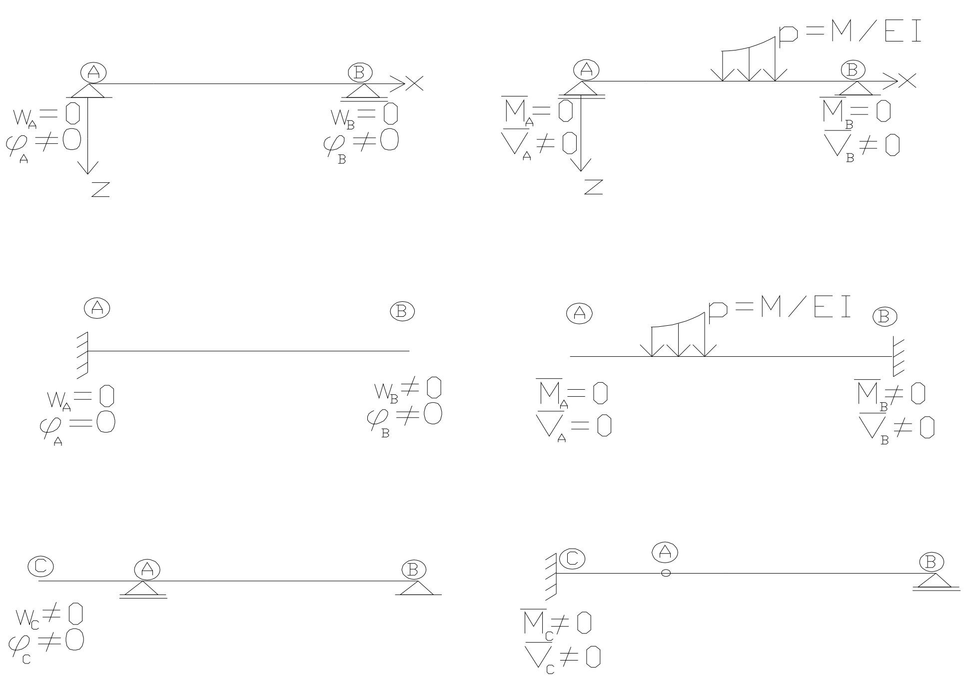
The external load p with $\frac{M}{EI}$

This load $p = \frac{M}{EI}$ will action upon a fictitious beam, called ***conjugate beam.*** Following the analogy made before, we may define:

*The **conjugate beam** is a fictitious beam which accomplish in stresses (bending moment $\overline{M}$ and shear force $\overline{V}$) the conditions accomplished by the real beam, in displacements (deflection w and rotation φ).*

Corresponding real and conjugate beams are shown below:

Real beam **Conjugate beam**



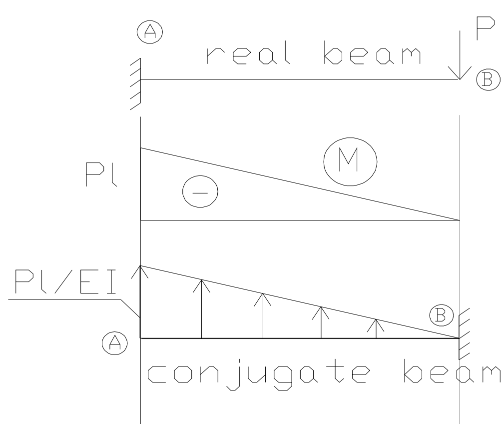
From the above comparisons, we can state two theorems related to the ***conjugate beam:***

Theorem 1: The rotation (slope) φ at a point in the real beam is numerically equal to the shear force $\overline{V}$ at the corresponding point in the conjugate beam.

Theorem 2: The displacement w (or v) of a point in the real beam is numerically equal to the bending moment $\overline{M}$ at the corresponding point in the conjugate beam

We shall exemplify the method for the same examples illustrated in the previous paragraph:

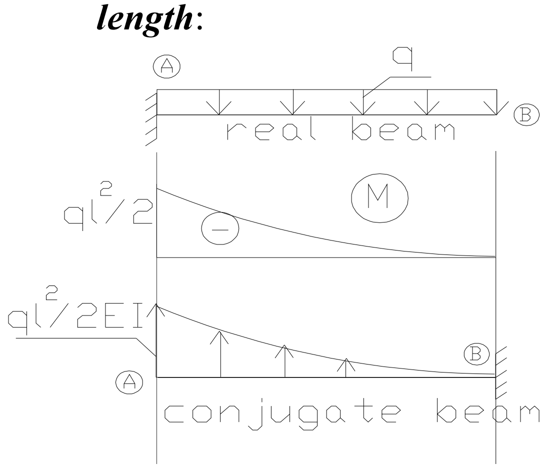
For the cantilever loaded by the concentrate force P acting in the free end:



$$w_{max} = w_B = \overline{M}_B = \frac{1}{2}\frac{Pl}{EI}l \cdot \frac{2}{3}l = \frac{Pl^3}{3EI}$$

$$\varphi_{max} = \varphi_B = \overline{V}_B = \frac{1}{2}\frac{Pl}{EI}l = \frac{Pl^2}{2EI}$$

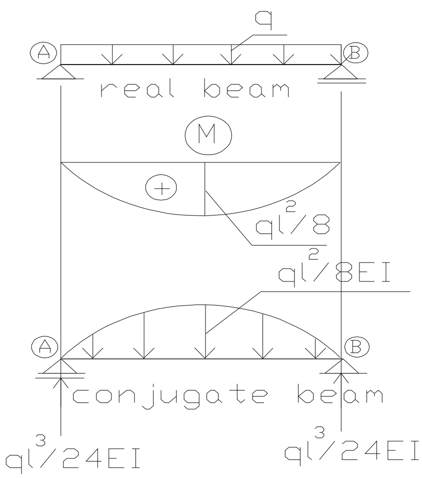
For the cantilever loaded by the uniformly distributed force q on the entire length:



$$w_{max} = w_B = \overline{M}_B = \frac{1}{3}\frac{ql^2}{2EI}l \cdot \frac{3}{4}l = \frac{ql^4}{8EI}$$

$$\varphi_{max} = \varphi_B = \overline{V}_B = \frac{1}{3}\frac{ql^2}{2EI}l = \frac{ql^3}{6EI}$$

For the simple supported beam loaded by the uniformly distributed force q on the entire length:



$$w_{max} = w\left(\frac{l}{2}\right) = \overline{M}\left(\frac{l}{2}\right) = \frac{2}{3}\cdot\frac{ql^2}{8EI}\cdot\frac{l}{2}\cdot\frac{l}{2} - \frac{ql^3}{24EI}\cdot\frac{3}{8}\cdot\frac{l}{2} = \frac{5ql^4}{384EI}$$

$$\varphi_{max} = \varphi(0) = \overline{V}(0) = \frac{ql^3}{24EI}$$

7.5 APPLICATIONS TO UNIAXIAL BENDING WITH SHEARING

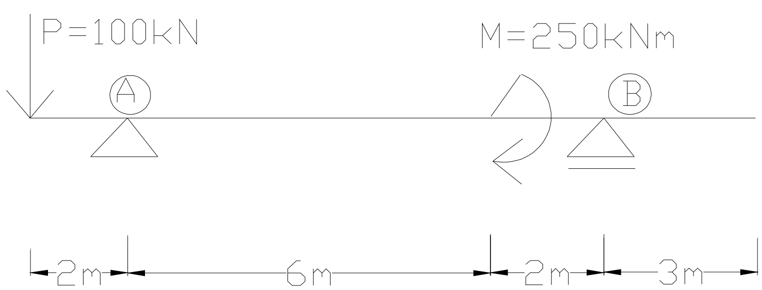
7.5.1 For the simple supported beam with the static scheme and the cross section from the figure bellow calculate:

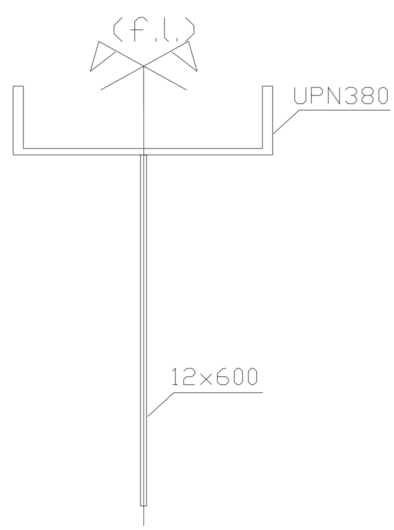
a. geometrical characteristics

b. diagrams of stresses

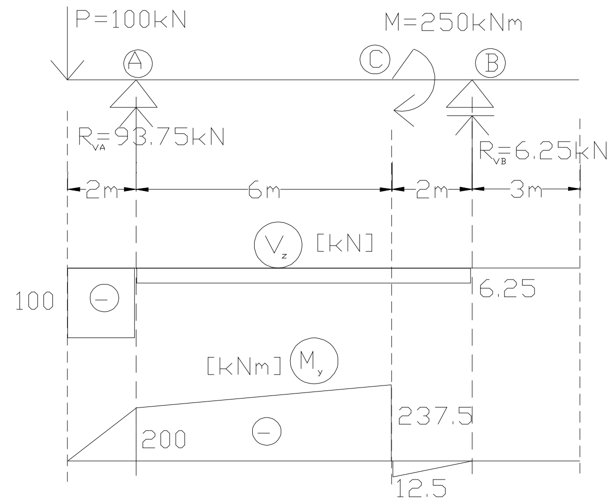
c. the strength verification at bending ($\sigma_{x\,max} = 2100 daN/cm^2$) and the normal stress diagram σ_x

d. in section A' (A left) the shear stress diagram τ_x with the significant values





The diagrams of stresses:

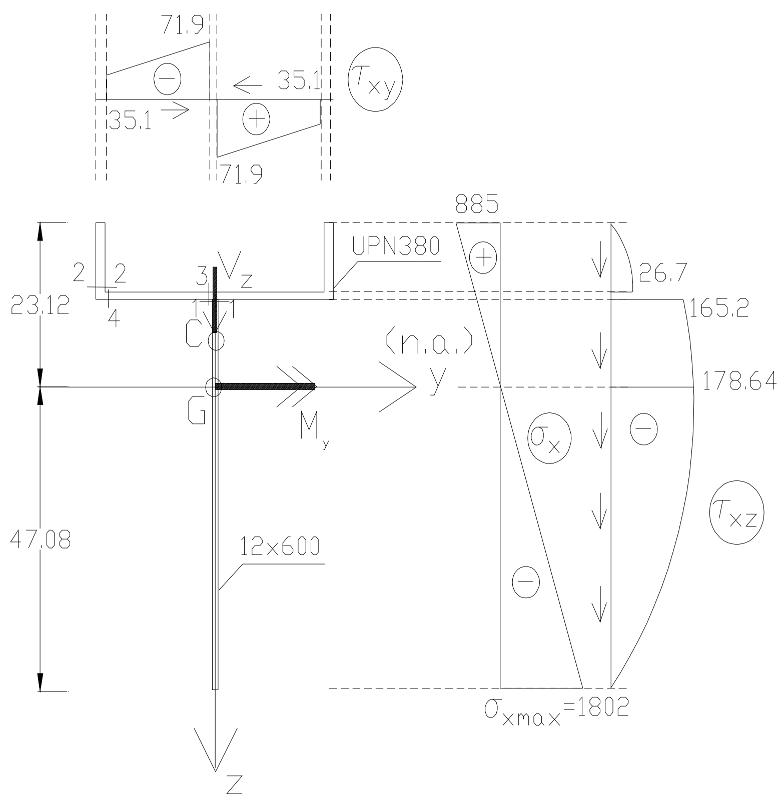


The critical section at bending is section C' (C left), where $M_{y\,max}$ = -237.5kNm.

With the second moment of area I_y = 62040cm^4, the maximum normal stress:

$$\sigma_{x\,max} = \frac{M_{y\,max}}{I_y} z_{max} = \frac{(-237.5 \times 10^4)}{62040} 47.08 = 1802 \frac{daN}{cm^2} < 2100 \frac{daN}{cm^2}$$

$$\sigma_{x\,min} = \frac{(-237.5 \times 10^4)}{62040} (-23.12) = 885 \frac{daN}{cm^2}$$



In section A' the shear force $V_{z\,max} = -100kN$

$$\tau_{xz\,max} = \frac{V_z S_y(z)}{b_z I_y} = \frac{100 \cdot 10^2 (47.08 \cdot 1.2 \cdot {}^{47.08}/_{2})}{1.2 \cdot 62040} = 178.64 \frac{daN}{cm^2}$$

$$\tau_{xz\,1\text{-}1} = \frac{100 \cdot 10^2 (80.4 \cdot 15.3)}{1.2 \cdot 62040} = 165.2 \frac{daN}{cm^2}$$

$$\tau_{xz\,2\text{-}2} = \frac{100 \cdot 10^2 (8.85 \cdot 1.6 \cdot 18.695)}{1.6 \cdot 62040} = 26.7 \frac{daN}{cm^2}$$

$$\tau_{xy\,3\text{-}3} = \frac{100 \cdot 10^2 (10.2 \cdot 1.6 \cdot 18.02 + 16.8 \cdot 1.35 \cdot 13.595)}{1.35 \cdot 62040} = 71.9 \frac{daN}{cm^2}$$

$$\tau_{xy\,4\text{-}4} = \frac{100 \cdot 10^2 (10.2 \cdot 1.6 \cdot 18.02)}{1.35 \cdot 62040} = 35.1 \frac{daN}{cm^2}$$

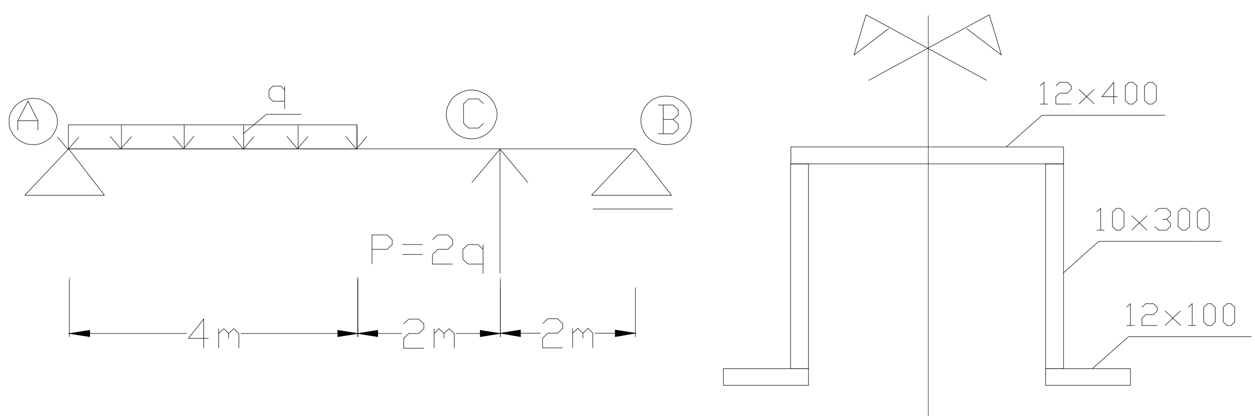
7.5.2 For the simple supported beam with the static scheme and the cross section from the figure bellow calculate:

a. geometrical characteristics

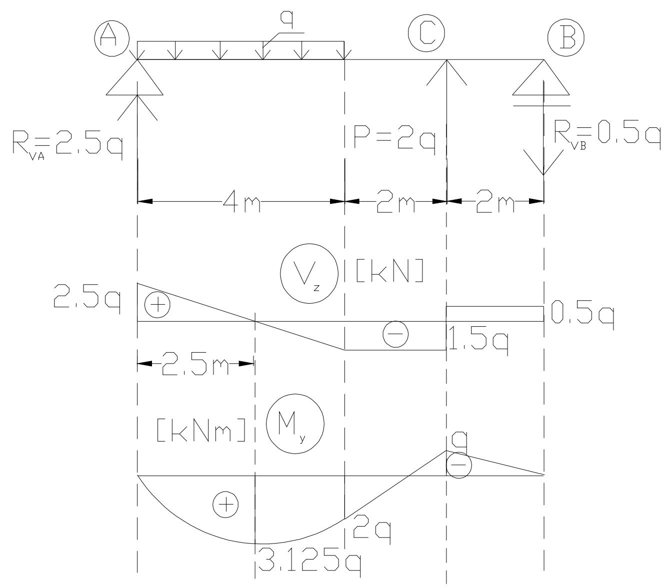
b. diagrams of stresses (function the load parameter q)

c. the load parameter q if $\sigma_{x\,max} = 2200 daN/cm^2$ and then the normal stress diagram σ_x

d. in section A'' (A right) the shear stress diagram τ_x with the significant values



The diagrams of stresses (function the load parameter *q*):



The critical section at bending is where $M_{y\,max} = 3.125q$

With the second moment of area $I_y = 20969cm^4$, the maximum normal stress:

$$\sigma_{x\,max} = \frac{M_{y\,max}}{I_y} z_{max} = \frac{(3.125q \times 10^4)}{20969} 19.04 = 2200 \frac{daN}{cm^2}$$

From the above equation, ***q = 77.5kN/m***

$$\sigma_{x\,min} = \frac{(3.125 \cdot 77.5 \times 10^4)}{20969}(-13.36) = -1543 \frac{daN}{cm^2}$$

In section A'' the shear force $V_{z\,max} = 193.75kN$

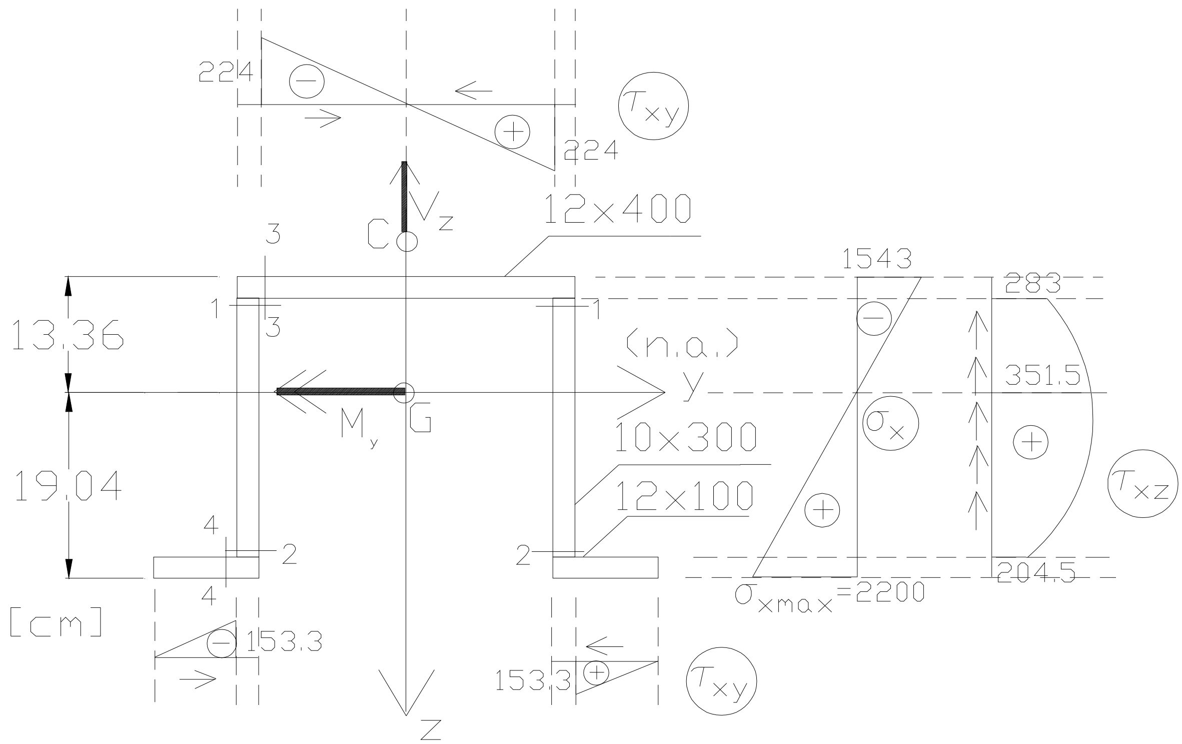
$$\tau_{xz\,max} = \frac{V_z S_y(z)}{b_z I_y} = \frac{193.75 \cdot 10^2 (17.84 \cdot 1 \cdot {}^{17.84}/_{2} + 1.2 \cdot 10 \cdot 18.44)}{1 \cdot 20969} = 351.5 \frac{daN}{cm^2}$$

$$\tau_{xz\,1\text{-}1} = \frac{193.75 \cdot 10^2 (1.2 \cdot 40 \cdot 12.76)}{2 \cdot 1 \cdot 20969} = 283 \frac{daN}{cm^2}$$

$$\tau_{xz\,2\text{-}2} = \frac{193.75 \cdot 10^2 (1.2 \cdot 10 \cdot 18.44)}{1 \cdot 20969} = 204.5 \frac{daN}{cm^2}$$

$$\tau_{xy\,3\text{-}3} = \frac{193.75 \cdot 10^2 (1.2 \cdot 19 \cdot 12.76)}{1.2 \cdot 20969} = 224 \frac{daN}{cm^2}$$

$$\tau_{xy\,4\text{-}4} = \frac{193.75 \cdot 10^2 (1.2 \cdot 9 \cdot 18.44)}{1.2 \cdot 20969} = 153.3 \frac{daN}{cm^2}$$


224
224
τxy
Vz
C
12×400
3
1
3
1
13.36
(n.a.)
y
My
G
19.04
10×300
12×100
4
2
2
4
[cm]
153.3
153.3
τxy
z
1543
283
351.5
σx
τxz
204.5
σxmax=2200