Chapter 3

THE STATIC ASPECT OF SOLICITATION

3.1. ACTIONS

Construction elements interact between them and with the environment. The consequence of this interaction defines the system of actions that subject the construction elements. This system is represented by forces, which shall be named **actions, loads.**

There are 3 main types of actions:

- **Permanent** or **dead loads** (G) are loads relatively constant in time, transferred to structure throughout the life span. E.g.: self-weight of the structural members, fixed permanent equipment or weight of different materials.
- **Variable** or **live loads** (Q) are temporary, of short duration. E.g.: imposed load, snow, wind, or temperature variation.
- **Accidental** (A) are loads lasting a few seconds. E.g.: earthquakes, explosions or impact of a vehicle.

All these actions must be **combined** in some manner. It would be unreasonable to assume that the maximum values of all loads occurred simultaneously.

The loads acting on constructions elements are:

a) **concentrated (point) loads** are single forces acting over a relatively small area: $P\downarrow$, $\swarrow Q$, $\overleftarrow{H}$, measured in [force] units: [kN], [daN], [t], [kg]…..

b) **distributed loads: p, q**
   - linear distributed loads, are loads that act along a line → on bars
   - surface distributed loads, acting over a surface area → on slabs
   - volume distributed loads (mass forces)

In calculations, the distributed load can be equated by a concentrated force, called **resultant.**

The **resultant R** (Fig. 3.1) is the **area** of the load diagram, and it is located in the center of gravity (centroid, center of mass) of this diagram.

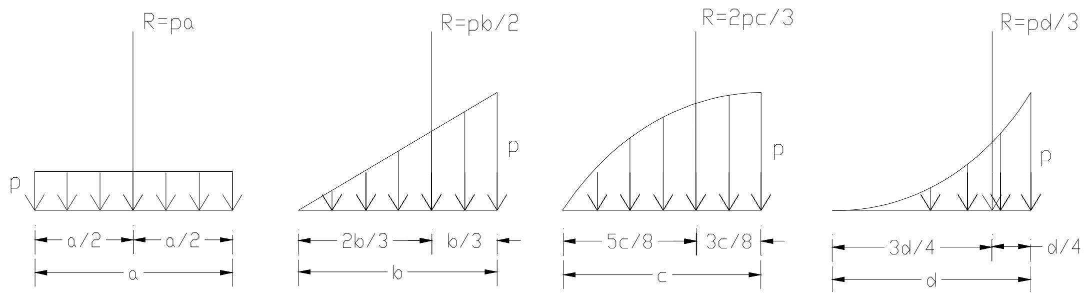


Fig. 3.1

- on bars: p [force/length] units: [kN/m], [daN/cm]…..
- on slabs: p [force/area] units: [kN/m^2], [daN/cm^2]…..
- in massifs : p [force/volume] units: [kN/m^3], [daN/cm^3]…..

c) **concentrated (point) moments:** M [force × length]

3.2 Supports

a) **The roller** (simple support)

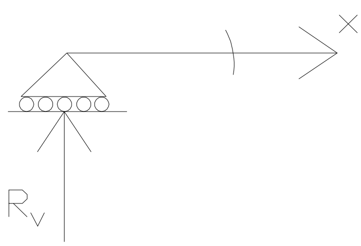


Fig. 3.2

- It is free to rotate and translate along the surface upon which the roller rests

- The translation perpendicular to the surface upon which the roller rests is prevented (Fig. 3.2)

- The reaction force $\mathbf{R_V}$ is always a single force that is perpendicular to the surface upon which the roller rests

b) **The pinned support** (hinge)

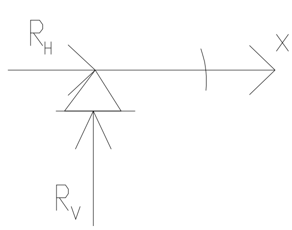


Fig. 3.3

- It will allow the structural member to rotate, but not to translate in any direction (Fig. 3.3)

- The reaction forces $\mathbf{R_V}$ and $\mathbf{R_H}$ are always two forces acting perpendicular, respectively tangent to the surface upon which the hinge rests.

c) **The fixed** (built-in) **support**

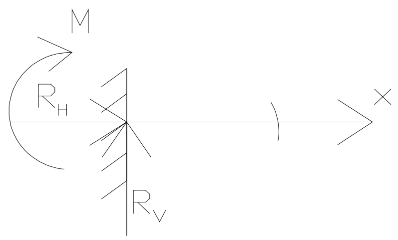


Fig. 3.4

- A fixed support does not permit rotation and translations in any direction

- There are 3 reactions: vertical and horizontal reactions $\mathbf{R_V}$ and $\mathbf{R_H}$ and a fixed end moment **M (**Fig. 3.4)

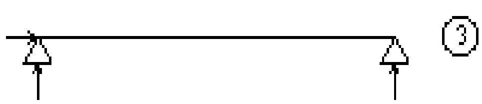
All construction elements should be in ***equilibrium*** *under the external actions and reactions from supports.*

In plan we may write 3 equations of equilibrium:

- *Vertical equilibrium*: total forces acting upward = total forces acting downward
- *Horizontal equilibrium*: total forces acting to the right = total forces acting to the left
- *Moment equilibrium*: total moments rotating clockwise = total moments rotating anticlockwise

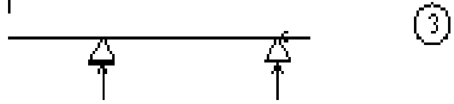
Having three available equations of equilibrium, we may solve a system which has up to three unknown reactions. That's why from **statically** point of view we met:

- **statically determinate structures** (isostatic):

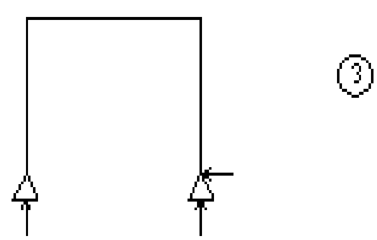


Simply supported beam

Cantilever



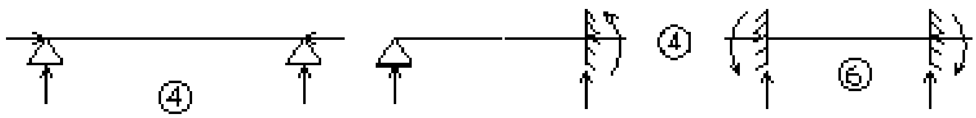
Simple supported beam with free ends (overhanging beams)



Simply supported frame

The number of unknown reactions is exactly equal to the number of available equations of equilibrium, three ($\sum H = 0$; $\sum V = 0$; $\sum M = 0$)

- **statically indeterminate structures** (hyperstatic): the number of the reactions from supports is greater than the number of the equilibrium equations.



- **unstable structures**: the number of the reactions from supports is smaller than the number of the equations of equilibrium.

3.3 REACTIONS

Considering the hypothesis of the small deformations, the equations of equilibrium may be written on the initial undistorted, undeformed member, acting like a rigid body (Fig. 3.5).

The 3 equations which will be used are: $\sum H = 0$; $\sum V = 0$; $\sum M = 0$

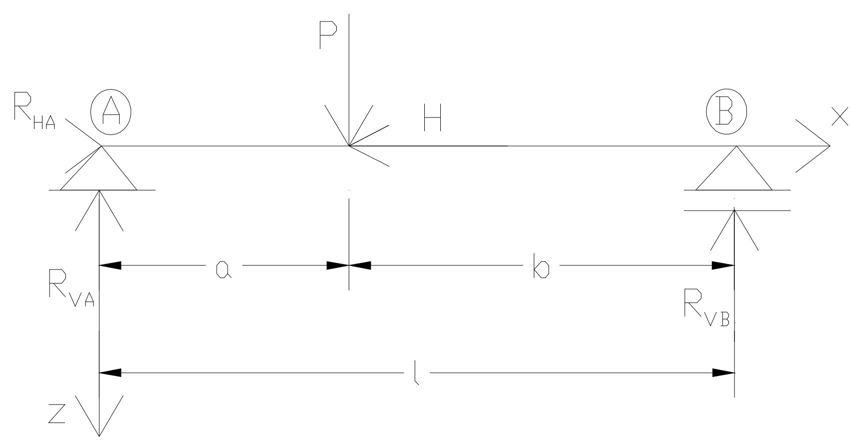


Fig. 3.5

$$\sum H = 0 \rightarrow \sum x = 0: R_{HA} - H = 0 => R_{HA} = H$$

$\overrightarrow{\oplus}$ (Forces acting to the right are positive)

$$\left(\sum M\right)_A = 0 \rightarrow P \cdot a - R_{VB} \cdot l = 0 => R_{VB} = P \cdot \frac{a}{l}$$

$\oplus\curvearrowright$ (The clockwise moment is arbitrarily taken as positive)

$$\left(\sum M\right)_B = 0 \rightarrow R_{VA} \cdot l - P \cdot b = 0 => R_{VA} = P \cdot \frac{b}{l}$$

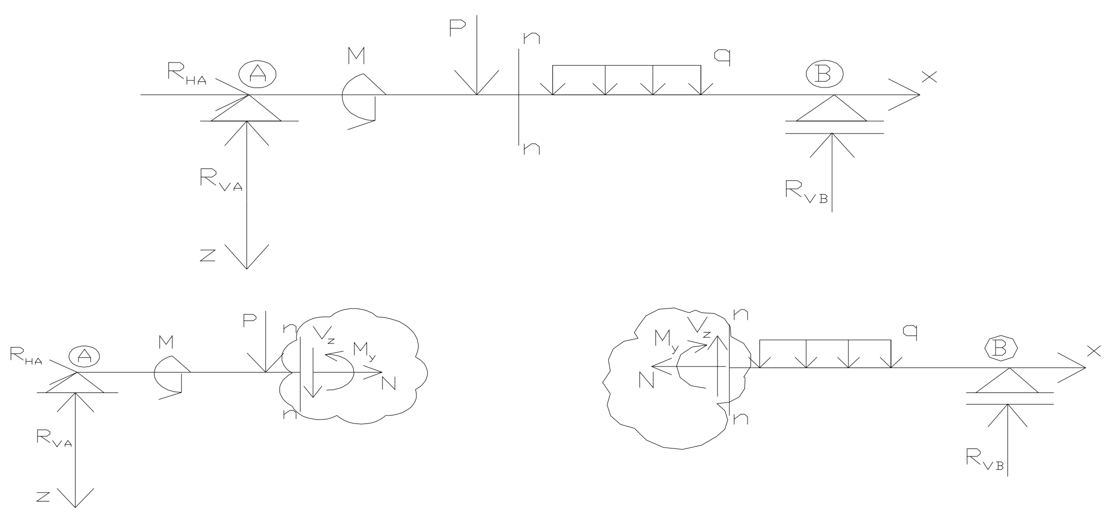
Verification:

$$\sum V = 0 \rightarrow \sum z = 0: \; - R_{VA} + P - R_{VB} = 0 \qquad \surd \; OK$$

$\downarrow\oplus$ (Forces acting downward are positive)

3.4 THE STATIC DEFINITIONS OF INTERNAL STRESSES (INTERNAL CONCENTRATED FORCES AND MOMENTS)

We consider a beam in static equilibrium under the action of external forces and reactions.



Left part (I) Right part (II)

Fig. 3.6

We may find the internal stresses, using the ***method of sections.*** We cut the bar (Fig. 3.6) with a section *n-n* perpendicular to the longitudinal bar axis Gx, obtaining 2 separate parts: left part (I) and right part (II).

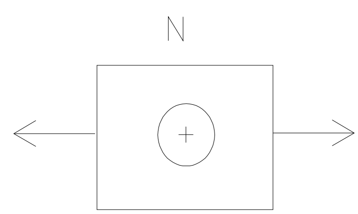
Assuming the left part is removed, to assure the equilibrium of the right part, on the face of this part we have to introduce ***internal stresses*** that replace the action of the removed part (left). These internal stresses are reduced in the cross section centroid (of the right part) to:

- an axial force N (tangent to the longitudinal bar axis)
- a shear force V_z (perpendicular to the longitudinal bar axis)
- a bending moment M_y

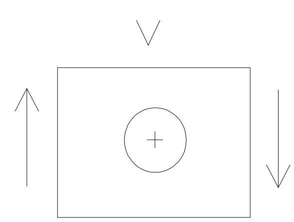
If instead the left part I, the right part II is removed, the internal stresses are introduced on the left part in the same manner, but opposite to the first case.

The ***internal stresses*** were introduced on each face with the positive convention. Let's define these stresses and see their positive signs:

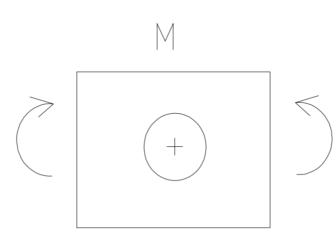
Axial force N: is the algebraic sum of all forces acting longitudinal (parallel) to the bar axis, taken from the left **or** the right side, from the section considered. It is positive when it produces tension in the cross section.



Shear force V: is the algebraic sum of all forces acting transverse (perpendicular) to the bar axis, taken from the left **or** the right side, from the section considered. It is positive when on the left it acts upward and contrary.



Bending moment M: is the algebraic sum of the moments of all forces taken from the left **or** the right side, from the section considered, taken about the centroid of the cross section. It is positive when it produces tension in the bottom fiber of the cross section.

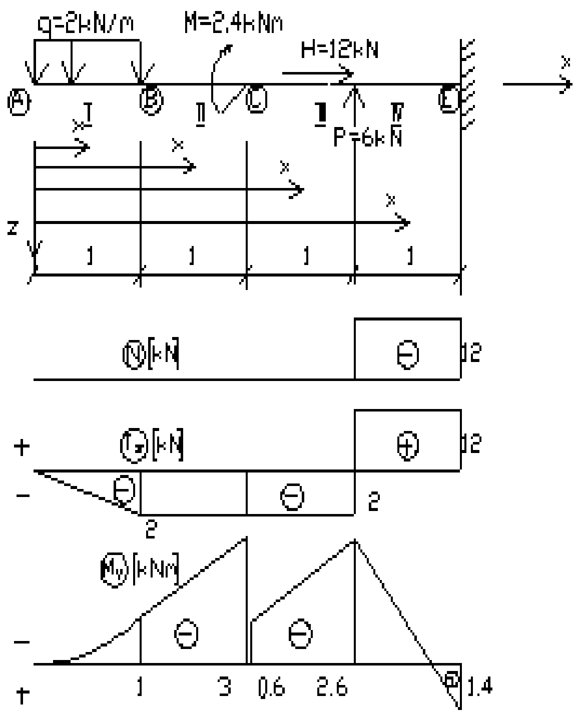


3.5. DIAGRAMS OF STRESSES

Each cross section is positioned along the bar by the variable *x*. The internal stresses are also functions of this variable *x*. The diagrams of stresses are even the graphs of these functions.

To draw these diagrams it is enough to write the analytical expression of N(x), $V_z(x)$ and $M_y(x)$ and then to represent the graphs of these functions. But, this procedure must be applied on different intervals, where external loads or reactions act.

Example:



A-B: $0 \leq x \leq 1$

N(x)=0

$\oplus\uparrow\ V_z(x) = -q \cdot x = -2x$

$\oplus\curvearrowleft\ M_y(x) = -q \cdot x \cdot \frac{x}{2} = -x^2$

B-C: $1 \leq x \leq 2$

N(x)=0

$V_z(x) = -q \cdot 1 = -2$

$M_y(x) = -q \cdot 1(x - 0{,}5) = -2x + 1$

C-D: $2 \leq x \leq 3$

N(x)=0

$V_z(x) = -q \cdot 1 = -2$

$M_y(x) = -q(x - 0{,}5) + M = -2x + 3{,}4$

D-E: $3 \leq x \leq 4$

$N(x) = -H = -12kN$

$V_z(x) = -q \cdot 1 + P = 4kN$

$M_y(x) = -q(x - 0{,}5) + M + P(x - 3) = 4x - 14{,}6$

3.6. RELATIONSHIPS BETWEEN LOADS AND STRESSES

We consider a differential element *dx* along the length of a beam (Fig.3.7). The ends of the element are subject to shear forces and moments which are represented with their positive directions. On this differential element a uniformly distributed load p_n acts, considered positive when it acts downward.

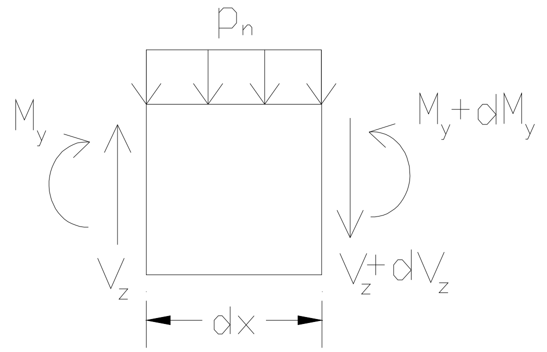


Fig.3.7

We write the ***vertical equilibrium***:

$$-V_z + p_n \cdot d_x + V_z + dV_z = 0$$

$$\boxed{\frac{dV_z}{dx} = -p_n} \quad (3.1)$$

The above relation (3.1) represents the first differential relationship between the shear force and load and it can be enunciated as: *the rate of change of shear force, which is the slope of the shear diagram, is minus the intensity of the distributed load acting transverse to x axis.*

Similarly (Fig.3.8), from ***horizontal equilibrium***:

$$-N + p_t \cdot d_x + N + dN = 0$$

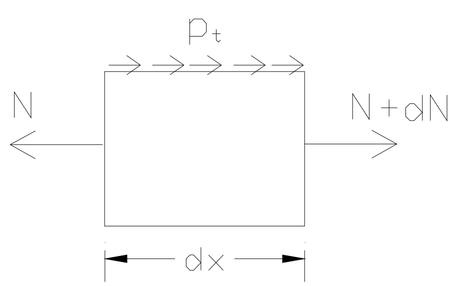


Fig.3.8

$$\boxed{\frac{dN}{dx} = -p_t} \quad (3.2)$$

This relation (3.2) represents the second differential relationship between the axial force and load and it can be enunciated as: *the rate of change (the first derivate) of axial force, which is the slope of the axial force diagram, is minus the intensity of the distributed load acting along x axis.*

Writing now the **moment equilibrium** about the centroid of the differential element (Fig.3.7):

$$M_y + V_z \cdot \frac{dx}{2} + V_z \cdot \frac{dx}{2} + dV_z \cdot \frac{dx}{2} - M_y - dM_y = 0$$

Neglecting the second order term, as dx →0 => $V_z \cdot dx = dM_y$ =>

$$\boxed{\frac{dM_y}{dx} = V_z} \quad (3.3)$$

This final relation (3.3) represents the third differential relationship between the bending moment and shear force and it can be enunciated as: *the rate of moment, which is the slope of the moment diagram, is equal to the shear force.*

These relations (first and last) may be written also:

$$\frac{d^2M_y}{dx^2} = \frac{dV_z}{dx} = -p_n \qquad (3.4)$$

To understand better these differential relations we may say that if $p_n(x)$ is a polynomial of degree "n", $V_z(x)$ is a polynomial of degree "n+1", respectively $M_y(x)$ is a polynomial of degree "n+2" (Fig.3.9).

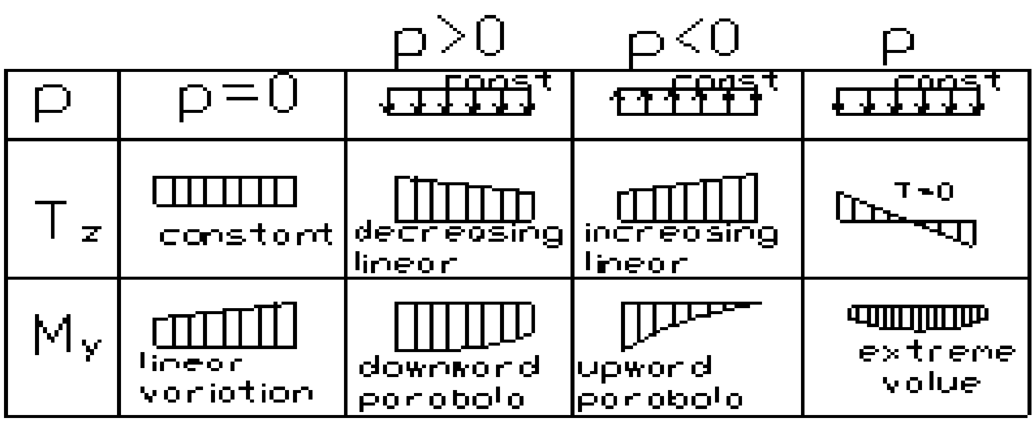


Fig.3.9

3.7. UNIT STRESSES

In the previous paragraphs we have seen that to determine the internal forces, the procedure to separate parts of the body, to exteriorize internal forces and to write equilibrium equations, are the main procedure in Mechanics of Materials. The internal forces in the sectioned plane represent the resultant of small contact forces that were released due to sectioning of that element (Fig.3.10).

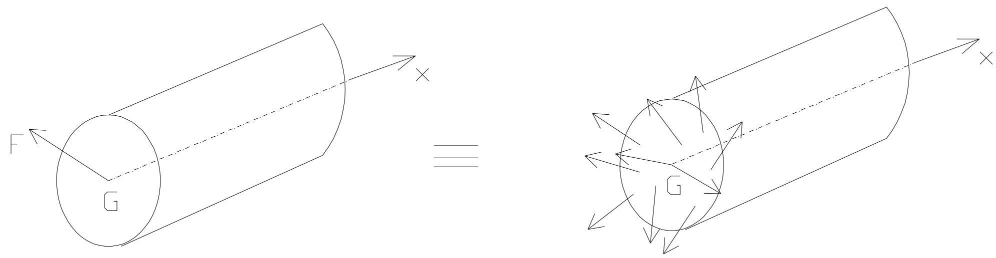


Fig.3.10

If an infinitely small area *dA* (Fig.3.11) is considered on the sectioned surface and the resultant force $d\vec{F}$ is calculated for this area, it can be assumed that:

(1) $d\vec{F}$ is applied in the centroid of *dA*

(2) if *dA* is very small, $d\vec{F}$ can be considered uniformly distributed on *dA*.

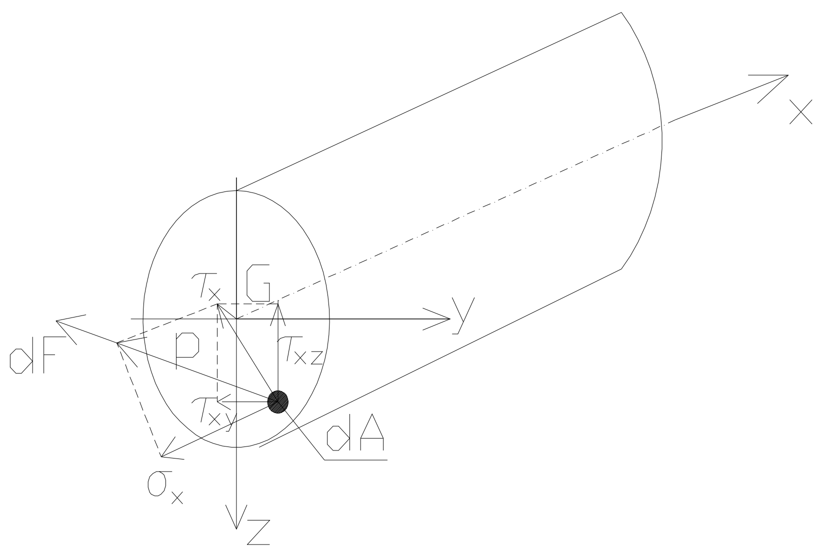


Fig.3.11

The distributed force on *dA* is:

$$\boxed{\vec{p} = \frac{d\vec{F}}{dA}} \qquad (3.5)$$

The new notion $\vec{p}$ is called ***unit stress***. If dA=1=> $\vec{p} = \mathrm{d}\vec{F}$, so the ***unit stress*** is ***the interior force uniformly transmitted through the unit surface*** (area). It is measured in $\left[\frac{force}{area}\right]$ units:$\left[\frac{kN}{m^2}\right]$, $\left[\frac{daN}{cm^2}\right]$, $[MPa]$.

The unit stress $\vec{p}$ is a vectorial notion, which can be decomposed in 2 main components (Fig.3.11):

- a normal component $\boldsymbol{\sigma_x}$, called ***normal unit stress*** or ***direct stress***
- a tangent component $\boldsymbol{\tau_x}$, called ***tangential unit stress*** or ***shear stress***

We observe that $p^2 = {\sigma_x}^2 + {\tau_x}^2$ and both components have a first index which indicate the direction of the axis which is perpendicular to the cross section.

The tangential stress τ_x can be decomposed (Fig.3.11) into 2 components: τ_{xz} parallel to *Gz* axis and τ_{xy} parallel to *Gy* axis.

The orientation of σ and τ from figure (Fig.3.11) is the positive one: σ_x is positive if it comes from the cross section.

3.8 THE STRENGTH CALCULATION OF STRESSES

(THE REDUCTION OF THE UNIT STRESSES IN THE CROSS SECTION)

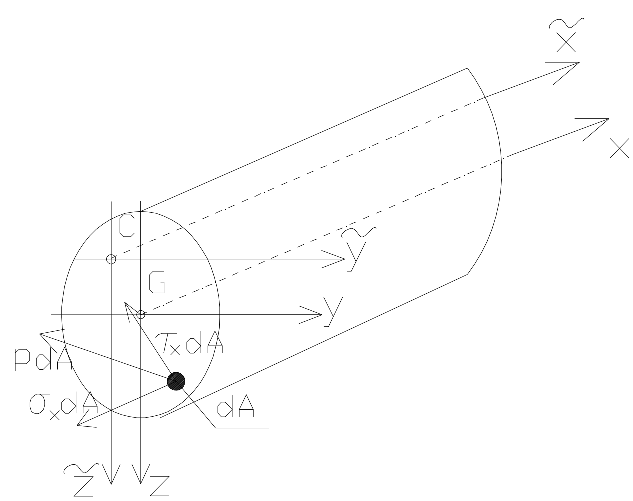


Fig.3.12

The resultant of the unit stress *p* on the differential area *dA* , *p·dA* , must be reduced into two significant points of the cross section (Fig.3.12):

- ***the center of gravity (centroid) G***, where the resultant of the normal stress $\sigma_x \cdot dA$ will be reduced
- ***the shear center C***, where the resultant of the tangential stress $\tau_x \cdot dA$ will be reduced

3.8.1 The reduction of the normal stress σ_x

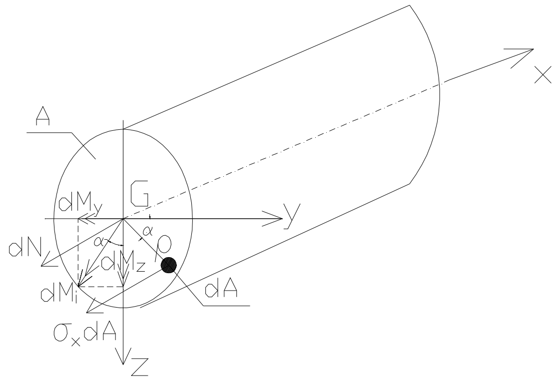


Fig.3.13

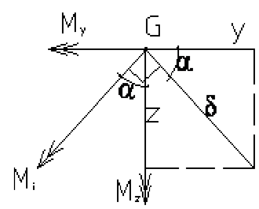
Reducing ($\sigma_x \cdot dA$) in *G* (Fig.3.13), we obtain:

- an elementary force:

$$dN = \sigma_x \cdot dA$$

- an elementary moment:

$$dM_i = (\sigma_x \cdot dA) \cdot \rho$$



$$y = \rho \cdot \cos\alpha$$

$$z = \rho \cdot \sin\alpha$$

The elementary moment dM_i is decomposed into:

$$dM_y = dM_i \cdot sin\alpha = (\sigma_x \cdot dA) \cdot \rho \cdot sin\alpha$$

$$dM_z = dM_i \cdot cos\alpha = (\sigma_x \cdot dA) \cdot \rho \cdot cos\alpha$$

So, the elementary components of the moment dM_i , about *Gy* and *Gz* axis are:

$$dM_y = \sigma_x \cdot z \cdot sin\alpha$$

$$dM_z = \sigma_x \cdot y \cdot cos\alpha$$

Extending all the elementary forces *dN* on the entire area *A,* we obtain finally ***the axial force*:**

$$\boxed{N = \int_A \sigma_x \cdot dA} \qquad (3.6)$$

Similarly, the elementary moments give ***the bending moments***:

$$\boxed{M_y = \int_A \sigma_x \cdot z \cdot dA} \qquad (3.7)$$

$$\boxed{M_z = \int_A \sigma_x \cdot y \cdot dA} \qquad (3.8)$$

!!! The bending moments have the vectors orientated along Gy and Gz axis (perpendicular to the longitudinal bar axis Gx).

The positive orientation (Fig.3.13) of the axial force N and the bending moments M_y and M_z on the cross section is:

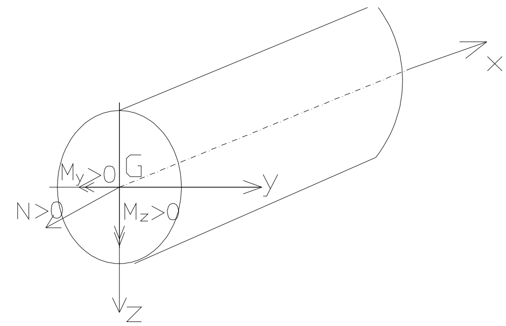


Fig.3.14

3.8.2 The reduction of the tangential stress τ_x

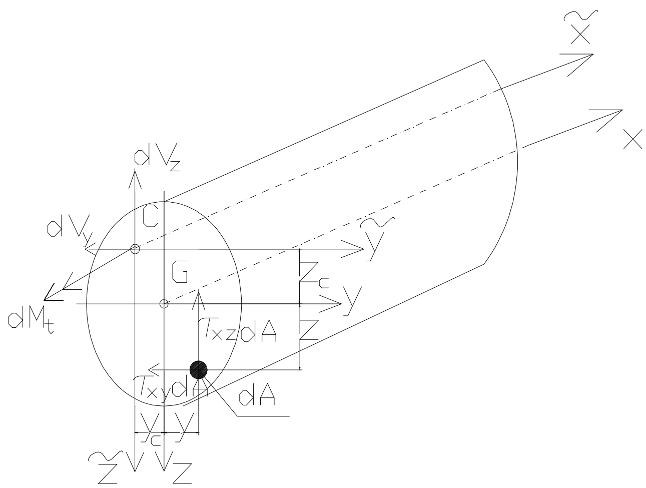


Fig.3.15

Reducing $(\tau_x \cdot dA)$ in *C* (Fig.3.15), through its components $(\tau_{xz} \cdot dA)$ $(\tau_{xy} \cdot dA)$, we obtain:

- the elementary forces:

$$dV_z = \tau_{xz} \cdot dA$$

$$dV_y = \tau_{xy} \cdot dA$$

- the elementary moment:

$$dM_t = dM_{t1} - dM_{t2} = \tau_{xz} \cdot dA(y - y_C) - \tau_{xz} \cdot dA(z - z_C)$$

Extending on the entire area *A*, we obtain ***the shear forces***:

$$\boxed{V_z = \int_A \tau_{xz} \cdot dA} \qquad (3.9)$$

$$\boxed{V_y = \int_A \tau_{xy} \cdot dA} \qquad (3.10)$$

Similarly, the elementary moment give ***the torsion moment (torque)***:

$$\boxed{M_t = \int_A [\tau_{xz}(\mathrm{y} - \mathrm{y_c}) - \tau_{xy}(\mathrm{z} - \mathrm{z_c})]\mathrm{dA}} \qquad (3.11)$$

!!!The torque M_t has the vector orientated along $\tilde{x}$ axis (the longitudinal axis of the shear center).

The positive orientation (Fig.3.15) of the shear forces V_z and V_y , and the torsion moment M_t on the cross section is:

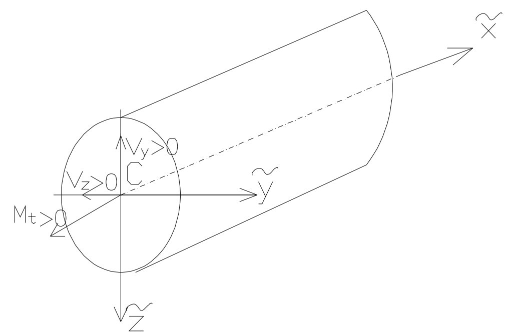


Fig.3.16

The internal stresses written in this way, from strength **calculation** is called the stress calculation **from interior**, while the **static calculation** is called calculation **from exterior.**

3.9. BARS ACTIONS (SOLICITATIONS)

a) Simple actions (when a single stress acts in the cross section)

- centric (axial) tension : $N>0$, ($V_z = V_y =0$, $M_y = M_z = M_t =0$)

- centric (axial) compression : $N<0$, ($V_z = V_y =0$, $M_y = M_z = M_t =0$)

- pure shearing : $V_z \neq 0$ or $V_y \neq 0$, ($N =0$, $M_y = M_z = M_t =0$)

- pure bending : $M_y \neq 0$ or $M_z \neq 0$, ($V_z = V_y =0$, $N = M_t =0$)

- pure torsion : $M_t \neq 0$, ($V_z = V_y =0$, $M_y = M_z = N =0$)

b) Compound actions (when more stresses act in the cross section)

- eccentric tension or compression : N and M_y and/or M_z

- pure skew (oblique) bending : M_y and M_z

- shearing with torsion: V_z and/or V_y and M_t

- oblique bending with shearing : M_y and M_z, V_y and V_z

- eccentric tension or compression with shearing: N and M_y and/or M_z , V_y and V_z

- eccentric tension or compression with shearing and torsion: N and M_y and/or M_z, V_y and V_z and M_t

3.10 APPLICATIONS FOR DIAGRAMS OF STRESSES

3.10.1 Represent the diagrams of stresses for the simply supported beam from Fig.3.17

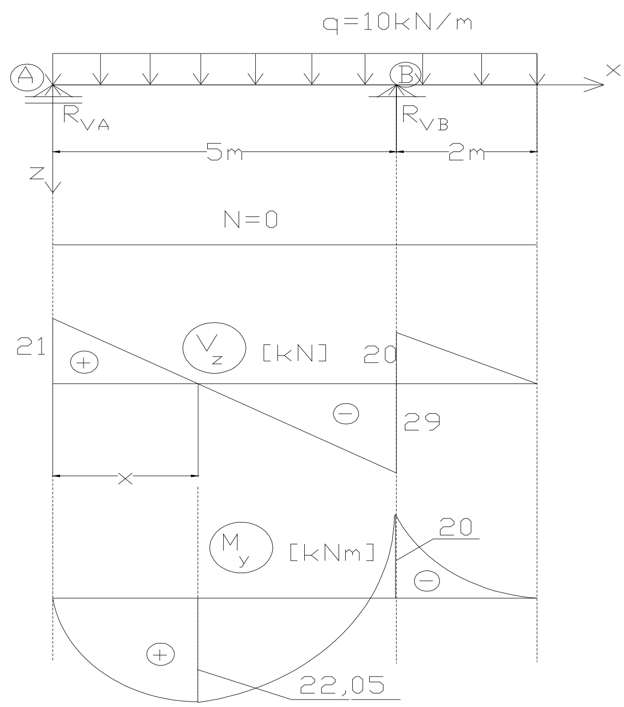


Fig.3.17

Reactions:

(1) $\sum x = 0\colon\ R_{HB} = 0$

(2) $(\sum M)_A = 0\colon\ q \cdot 7 \cdot 3{,}5 - R_{VB} \cdot 5 = 0\ \rightarrow R_{VB} = 49$ kN

(3) $(\sum M)_B = 0\colon\ -q \cdot 7 \cdot 1{,}5 + R_{VA} \cdot 5 = 0\ \rightarrow R_{VA} = 21$ kN

Verification:

$\sum z = 0\colon -R_{VA} - R_{VB} + q \cdot 7 = 0$ OK

Stresses:

Writing efforts will be traversing bar from left to right. In the sections " i " where concentrated forces act (horizontal and vertical), the axial force N and the shear force V will be written in sections located infinitely left " i' " and infinitely right " i'' " section, with respect to the section " i ". Bending moment M will be written directly into the section.

For a better understanding of writing efforts in a section, in this application efforts will be written in each section only taking the first left and then the right section, obviously obtaining the same results.

Axial force N

With no forces tangential or parallel to the longitudinal axis of the bar, axial force is zero throughout the bar.

Shear force V

$V_{A''} = R_{VA} = 21kN$ (taken from the left)

$V_{A''} = q \cdot 7 - R_{VB} = 21kN$ (taken from the right)

$V_{B'} = R_{VA} - q \cdot 5 = -29kN$ (taken from the left)

$V_{B'} = -R_{VB} + q \cdot 2 = -29kN$ (taken from the right)

$V_{B''} = R_{VA} - q \cdot 5 + R_{VB} = 20kN$ (taken from the left)

$V_{B''} = q \cdot 2 = 20kN$ (taken from the right)

Bending moment M

$M_A = 0$ (taken from the left)

$M_A = -q \cdot 7 \cdot 3{,}5 + R_{VB} \cdot 5 = 0$ (taken from the right)

$M_B = -q \cdot 5 \cdot 2{,}5 + R_{VA} \cdot 5 = -20\ kNm$ (taken from the left)

$M_B = -q \cdot 2 \cdot 1 = -20kNm$ (taken from the right)

In the free end V = M = 0.

Diagrams (graphs) of stresses:

- ***N* diagram** is represented with positive values (tensile stress) above the reference line, and negative values (compression) below the reference line

- ***V* diagram** is represented with positive values above the reference line, and negative values below the reference line

- ***M* diagram** is always represented on the tension fiber (for convenience, the positive values are below the reference line, and the negative values are above the reference line)

Returning to our problem, with the values obtained the diagrams of stresses are represented. It is noted that V is zero in a section (between A and B) whose position is unknown. Given the differential relationship (3.3) that exists between V and M, in the section where V is zero, M will have an extreme value. To calculate the extreme value, we must first determine where is this section of V=0. We denote the distance from the support A to that section with *"x"*, and we write the condition that in that section V (x) = 0.

$$V(x) = R_{VA} - q \cdot x = 0 \quad \rightarrow x = 2{,}1m$$

With this known distance, the maximum moment is:

$$M_{max} = M(x) = -q \cdot x \cdot \frac{x}{2} + R_{VA} \cdot x = 22{,}05\ kNm$$

From the graphs we see again the connection between the distributed load q, V and M, in that V is a high degree (linear variation) with respect to the uniformly distributed load q (constant), and M is with a high degree (parabolic variation) with respect to V, having a maximum section where V is zero. In the section where a concentrated force acts, V diagram will present a sudden jump, in the amount equal to the force.

3.10.2 Represent the diagrams of stresses for the simply supported beam from Fig.3.18

Reactions:

(1) $\sum x = 0: -F_1 - R_{HB} + F_2 = 0 \quad \rightarrow R_{HB} = 200$ kN

(2) $(\sum M)_A = 0:\ q \cdot 8 \cdot 1 - R_{VB} \cdot 7 + P \cdot 7 - M_0 = 0 \rightarrow R_{VB} = 160$ kN

(3) $(\sum M)_B = 0:\ - q \cdot 8 \cdot 6 + R_{VA} \cdot 7 - M_0 = 0 \rightarrow R_{VA} = 300$ kN

Verification:

$$\sum z = 0: -R_{VA} - R_{VB} + q \cdot 8 + P = 0 \qquad \text{OK}$$

Axial force N

$$N_{1''} = F_1 = 250kN = N_{2'}$$

$$N_{2''} = F_1 - F_2 \quad = 200kN = N_{B'}$$

$$N_{B''} = N_3 = 0$$

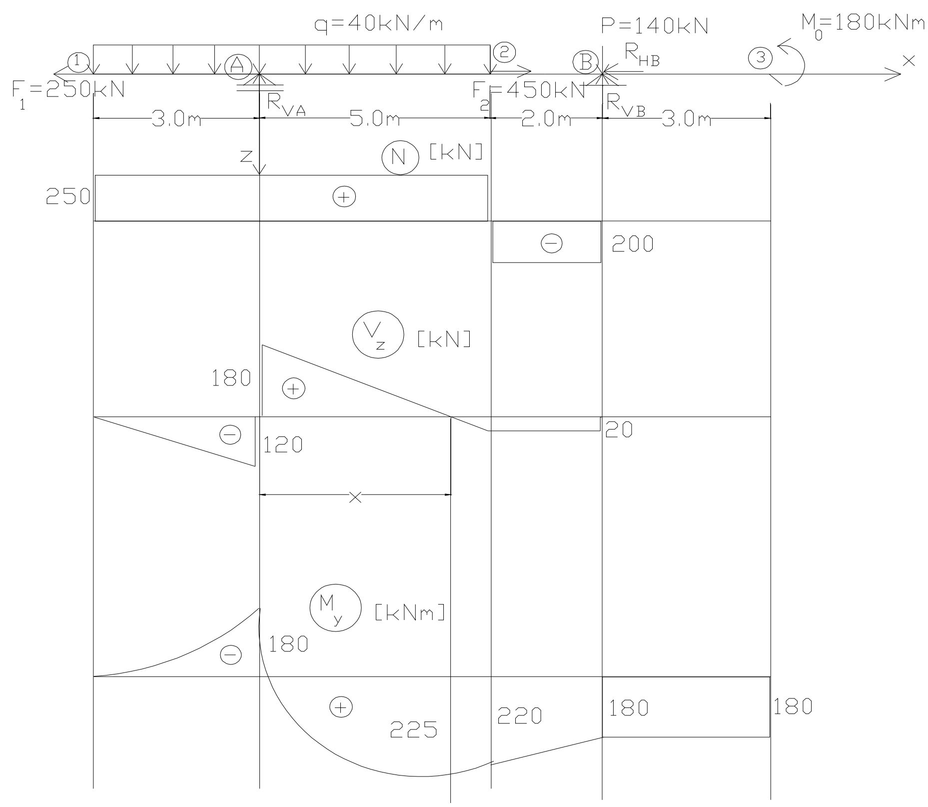


Fig.3.18

Shear force V_z

$$V_{1''} = 0$$

$$V_{A'} = -q \cdot 3 = -120kN$$

$$V_{A''} = -q \cdot 3 + R_{VA} = 180kN$$

$$V_2 = -q \cdot 8 + R_{VA} = -20kN \quad = V_{B'}$$

$$V_{B''} = 0 = V_{3'}$$

Bending moment M_y

$$M_{1''} = 0$$

$$M_A = -q \cdot 3 \cdot 1{,}5 = -180kNm$$

$$M_2 = -q \cdot 8 \cdot 4 + R_{VA} \cdot 5 = 220kNm$$

$$M_B = M_0 = 180\ kNm = M_3$$

Determine the position of the section in which V = 0:

$$V(x) = R_{VA} - q \cdot x = 0 \quad \rightarrow x = 7{,}5m$$

The maximum moment is:

$$M_{max} = M(x) = -q \cdot x \cdot \frac{x}{2} + R_{VA}(x-3) = 225\ kNm$$

3.10.3 Represent the diagrams of stresses for the cantilever from Fig.3.19

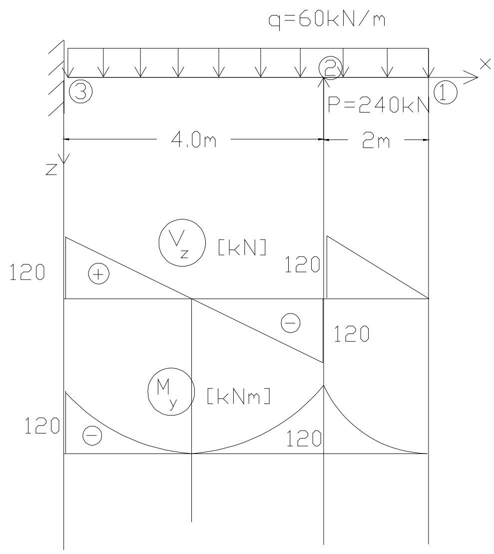


Fig.3.19

On cantilever we can write the stresses from the free end to the fixed support (in this application, from the right), such that the calculation of the reactions is not necessary. The axial force N is zero in the entire bar.

Shear force V

$$V_1 = 0$$

$$V_{2''} = q \cdot 2 = 120kN$$

$$V_{2'} = q \cdot 2 - P = -120kN$$

$$V_{A''} = q \cdot 6 - P = 120kN$$

Bending moment M

$$M_1 = 0$$

$$M_2 = -q \cdot 2 \cdot 1 = -120kNm$$

$$M_A = -q \cdot 6 \cdot 3 + P \cdot 4 = -120kNm$$

$$M_{max} = -q \cdot 4 \cdot 2 + P \cdot 2 = 0$$