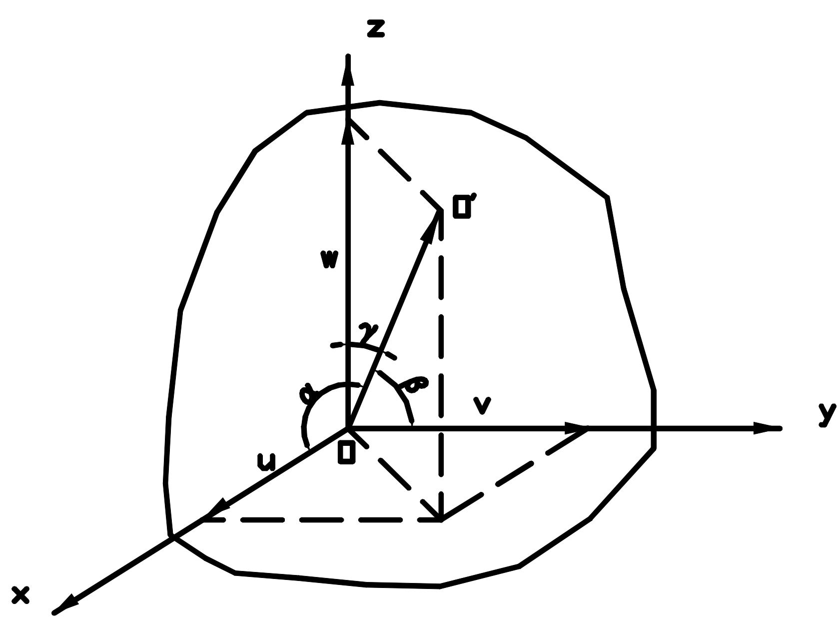
Chapter 2
GEOMETRIC ASPECT OF THE STATE OF SOLICITATION

2.1 THE DEFORMATION AROUND A POINT

2.1.1 The relative displacement

Due to the influence of external forces, temperature variation, magnetic and electric fields, the construction bodies are deformed and distorted (modifying their dimensions and shape). The geometric aspect studies this deformation as a geometrical phenomenon, produced by the relative displacement of the points from the studied body.
To observe the modifications of geometrical nature produced by external causes, we consider a body into an orthogonal rectangular system of axis, with the origin in O (Fig.2.1).



Due to the external forces that subject the body, point O reaches the position O'. The vector OO' is called *relative displacement* , and: $OO' = \vec{d}$

Fig.2.1

Projecting $\vec{d}$ on the axis *Ox*, *Oy* and *Oz*, we obtain the components of the relative displacements: on *Ox* axis it is noted with *u*, on *Oy* axis it is noted with *v* and on *Oz* axis it is noted with *w*.

From figure 2.1, we may write:

$$\cos\alpha = \frac{u}{\left|\vec{d}\right|}\ ,\ \cos\beta = \frac{v}{\left|\vec{d}\right|}\ ,\ \cos\gamma = \frac{w}{\left|\vec{d}\right|} \tag{2.1}$$

Squaring and then adding these relations, we get:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 = \frac{u^2 + v^2 + w^2}{\left|\vec{d}\right|^2} \tag{2.2}$$

The *relative displacement* is:

$$\left|\vec{d}\right| = \sqrt{u^2 + v^2 + w^2} \tag{2.3}$$

Function the ratio between the relative displacements and the initial dimensions of the body (for bars, the cross section dimensions), the problems from Structural Mechanics are studied with:

a. *The 1st order Theory:* if the relative displacements are very small with respect to the initial dimensions of the construction body. In this category are placed most of the problems from Mechanics of Materials. In this situation, the mode of supporting and loading of the body is not influenced by the structure deformation, being defined for the initial undeformed position of the structure.

b. *The 2nd order Theory:* if the relative displacements are comparable with the initial dimensions of the construction body.

The displacement of a point with respect to its initial position, called *relative displacement*, defined above, represents the *exterior aspect* of the deformation of a body. The *interior aspect* is given by the *modification of the volume and the shape* of the body, which is a complex deformation, but it can be illustrated by two simple deformations (strains):

- *the elongation* (linear strain)
- *the sliding* (angular strain)

2.1.2 The elongation

The elongation is a linear deformation which will be analyzed on an axis Ox (Fig.2.2.a). A segment from a deformable body is in the initial position AB. After deformation, this segment reaches the position A'B', due to the hypothesis of the continuity of displacements inside a body. The initial segment AB has the length l (Fig.2.2.a), which becomes $l+\Delta l$ for the segment A'B'. This increased length of the segment A'B' makes evident *the lengthening (elongation)* Δl of the initial segment AB. Therefore *the elongation* represents *the quantity with which the initial length of a segment is modified.* By convention Δl is *positive* if it is a *lengthening*, respectively *negative* if it is a *shortening.*

As the elongation Δl depends on the initial length of the segment, it isn't an adequate measure to characterize the linear deformation. That's why it is introduced a new notion called *specific elongation,* which represents *the total lengthening of the unit length* (that's why it is named also *unit elongation):*

$$\varepsilon_x = \frac{\Delta l}{l} \tag{2.4}$$

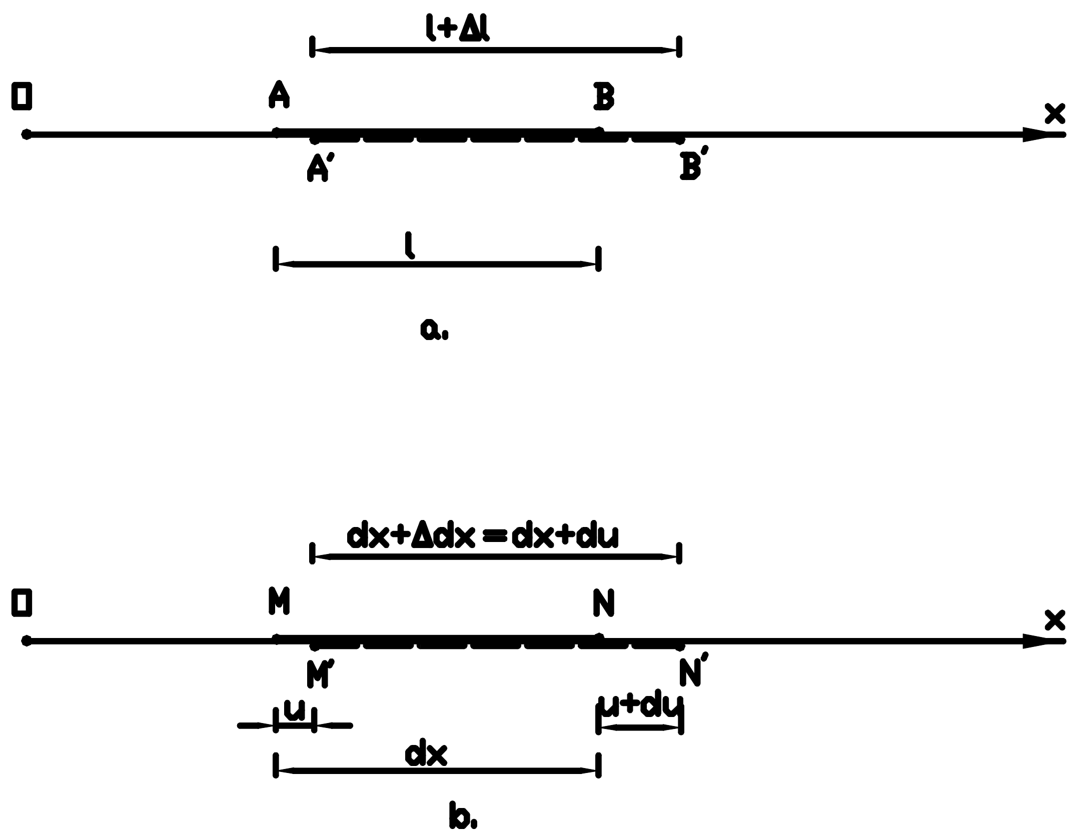


Fig.2.2

We observe that for the unit length l = 1, the specific elongation ε_x is even Δl*:*

$$(\varepsilon_x)_{l=1} = \Delta l \tag{2.5}$$

The specific elongation ε_x is a non-dimensional notion and the above relation is valid only if Δl is uniformly distributed along the entire length l, so only if ε_x = const. Hence, the total lengthening isn't produced by identical lengthening of each unit of length.

That's why we consider again a differential segment MN of length dx on the same axis Ox (Fig.2.2.b). The displacement of point M is u, but the one of point N will vary with the differential measure du, so the displacement of N will be $u+du$. The total elongation of the differential length dx will be:

$$\Delta dx = du \tag{2.6}$$

Considering that this total elongation is uniformly distributed on the differential length *dx*, the specific elongation will be now:

$$\varepsilon_x = \frac{\Delta dx}{dx} = \frac{du}{dx} \tag{2.7}$$

In the relation written above it was considered that *u* is function only of *x*. In reality it is function of all three coordinates $u = u(x,y,z)$ and the differential *du* has the general expression:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \tag{2.8}$$

But as the segment MN is on *Ox* axis, the single coordinate that varies is *x* and the other derivatives, with respect to *Oy* and *Oz* axis, are 0. So, *du* will be:

$$du = \frac{\partial u}{\partial x}dx \tag{2.9}$$

And ε_x is:

$$\varepsilon_x = \frac{du}{dx} = \frac{\frac{\partial u}{\partial x}dx}{dx} \quad \Rightarrow \quad \boxed{\varepsilon_x = \frac{\partial u}{\partial x}} \tag{2.10}$$

Similarly, the specific elongations along *Oy* and *Oz* axis may be written:

$$\boxed{\varepsilon_y = \frac{\partial v}{\partial y}} \tag{2.11}$$

and

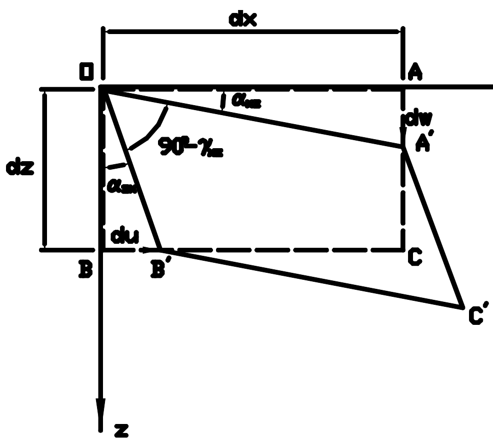
$$\boxed{\varepsilon_z = \frac{\partial w}{\partial z}} \tag{2.12}$$

Relations (2.10)…(2.12) are the first three *Cauchy's relations* representing the differential relations between the specific elongations and the components of the relative displacement.

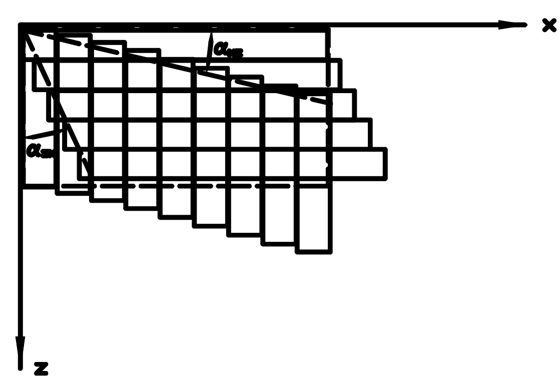
2.1.3 The sliding (shear strain)

Cause by the fact that the sliding is an angular deformation it will be analyzed in a plan from a deformable body. We consider a differential rectangle in a plan *xOz* , a corner of the rectangle being even the origin O (Fig.2.3.a). Admitting that point O is fixed, point A will reach a position perpendicular to *Ox* axis A', *AA' = dw* , respectively point B will reach a position perpendicular to *Oz* axis B', *BB' = du.* This deformation doesn't change the rectangle area, only its shape, transforming it into a parallelogram.

As a result of this deformation the initial straight angle *xOz* (Fig. 2.3.a,b) is modified, by the rotation of *Ox* axis with the angle α_{xz} and respectively *Oz* axis with α_{zx}.



a.



b.

Fig.2.3

Assuming the plan is made from strips parallel to *x* and *z* axis (Fig.2.3b), we observe that the above rotations are produced in fact by relative translations of the strips, called *slidings.* Hence, the angles α_{xz} and α_{zx} are a measure of these slidings. Their sum is the *specific sliding* γ_{xz} :

$$\gamma_{xz} = \alpha_{xz} + \alpha_{zx} \tag{2.13}$$

The *specific sliding* can be defined as *the modification of the initial straight angle.* The specific sliding (shear strain) is positive if decreases the initial straight angle (it becomes an acute angle), respectively negative if it increases this (becomes an obtuse angle).

Taking into account the hypothesis of the small deformations, the angles α_{xz} and α_{zx} may be written:

$$\alpha_{xz} \cong tg\alpha_{xz} = \frac{dw}{dx} = \frac{\frac{\partial w}{\partial x}dx}{dx} = \frac{\partial w}{\partial x} \tag{2.14}$$

and

$$\alpha_{zx} \cong tg\alpha_{zx} = \frac{du}{dz} = \frac{\frac{\partial u}{\partial z}dz}{dz} = \frac{\partial u}{\partial z} \tag{2.15}$$

Replacing in (2.13) we obtain:

$$\boxed{\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}} \tag{2.16}$$

Similarly, the specific slidings in *xOy* plan and *zOy* plan are:

$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}} \tag{2.17}$$

and

$$\boxed{\gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}} \tag{2.18}$$

Relations (2.16)…(2.18) are the other three *Cauchy's relations* representing the differential relations between the specific sliding and the components of the relative displacement.

Augustin-Louis Cauchy (1789-1857)

2.2 GEOMETRICAL CHARACTERISTICS OF THE CROSS SECTION

To use the notion of *cross section* in all calculations made in Mechanics of Materials, certain important characteristics should be known. These geometrical characteristics may be grouped, approximately, in two categories:

1. Characteristics that define the relative position of the system of reference *yOz* either it has an arbitrary origin, or identical to the cross section centroid G or the shear center C. In this category there are: *the area*, *the centroid G*, *the shear center C*, but also the *cross section shape* which is very important in the calculation of the construction members at certain solicitations.

2. Characteristics connected to the hypothesis of the movement of the cross section, which have also a mechanical interpretation. In this group the following characteristics are included: *the first moment of area (static moment): axial and sectorial, the second moments of area (moments of inertia): axial and sectorial, the radius of gyration (radius of inertia).*

2.2.1 The area. The first moment of the area. The centroid

We consider a certain cross section in an orthogonal system of axis $y_1O_1z_1$, with the origin in O_1 (Fig.2.4). The cross section of area A is made from infinite differential areas dA.

a. ***The area A*** is:

$$\boxed{A = \int_A dA} \tag{2.19}$$

Therefore, *the area* is the infinite sum of all the elementary areas *dA,* on the entire area *A*. It is always measured in [length]^2 units: cm^2 , m^2 , mm^2.

b. ***The first moment of the area (static moment)***:

$$\boxed{S_{\overline{y_1}} = \int_A \overline{z_1} dA} \qquad \boxed{S_{\overline{z_1}} = \int_A \overline{y_1} dA} \tag{2.20}$$

The static moment of an area with respect to an axis, is the infinite sum of all products between that area and the distance between that area centroid and the respective axis (Fig.2.4). The static moment can be positive or negative and it is always measured in [length]^3 units: cm^3 , m^3 , mm^3.

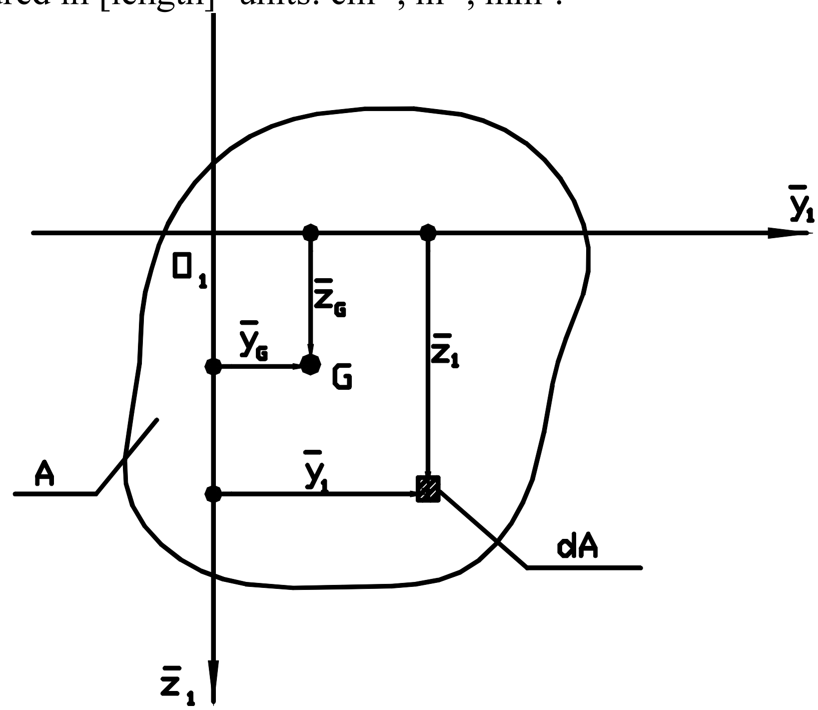


Fig. 2.4

c. ***The centroid (center of gravity)*** position is given by its coordinates $\overline{y_G}$ and $\overline{z_G}$ (Fig.2.4), where with G we note the centroid.

We write Varignon's theorem, which says that *the moment of the entire area is equal to the sum of all moments of the elementary areas:*

$$A \cdot \overline{y_G} = \int_A \overline{y_1} \cdot dA \qquad (2.21)$$

Pierre Varignon (1654-1722)

Respectively

$$A \cdot \overline{z_G} = \int_A \overline{z_1} \cdot dA \qquad (2.22)$$

From (2.21) and (2.22) the centroid coordinates are:

$$\boxed{\overline{y_G} = \frac{\int_A \overline{y_1} \cdot dA}{A} = \frac{S_{\overline{z_1}}}{A}} \qquad (2.23)$$

and

$$\boxed{\overline{z_G} = \frac{\int_A \overline{z_1} \cdot dA}{A} = \frac{S_{\overline{y_1}}}{A}} \qquad (2.24)$$

These coordinates are always measured in [length] units: cm, m, mm.

Observing relations (2.23) and (2.24) we may conclude that if the point O_1 is identically to the centroid G, $\overline{y_1} = \overline{z_1} = 0$ and implicit $\overline{y_G} = \overline{z_G} = 0$. This means that in this case the static moments are null $S_{\overline{y}} = S_{\overline{z}} = 0$, and these axes: $\overline{y}$ and $\overline{z}$ are called *central axis.*

!!! *If with respect to an axis the static moment is null, that axis is a central axis.*

If the cross section is made up of many sections having the area A_i and the centroid position G_i known (Fig.2.5), the integrals from (2.19), (2.20), (2.23) and (2.24) are transformed into finite sum:

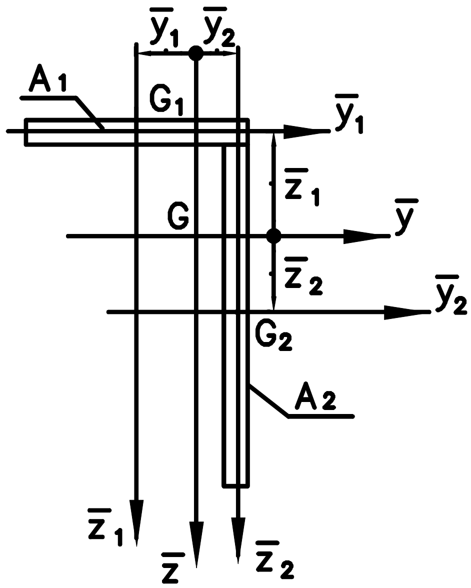


Fig.2.5

$$\boxed{A = \sum_{i=1}^{n} A_i} \quad (2.25)$$

$$\boxed{S_{\bar{y}} = \sum_{i=1}^{n} \overline{z}_i A_i} \quad (2.26)$$

$$\boxed{S_{\bar{z}} = \sum_{i=1}^{n} \overline{y}_i A_i} \quad (2.27)$$

$$\boxed{\overline{y_G} = \frac{\sum_{i=1}^{n} \overline{y}_i \cdot A_i}{A}} \quad (2.28)$$

$$\boxed{\overline{z_G} = \frac{\sum_{i=1}^{n} \overline{z}_i \cdot A_i}{A}} \quad (2.29)$$

2.2.2 The second moment of the area (area moment of inertia) and other geometrical characteristics

The area moment of inertia represents the cross section inertia to its movement of rotation around an axis included in its plane. The second moment of area is a measure of resistance to bending of a loaded section.
In engineering contexts, the area moment of inertia is often called simply *"the" moment of inertia* even though it is *not* equivalent to the usual moment of inertia (which has dimensions of mass times length squared and characterizes the angular acceleration undergone by a solids when subjected to a torque).

We may define:

a. ***The axial moment of inertia*** of an area A (Fig.2.6) with respect to an axis comprised in its plane represents the infinite sum of all products between the elementary area dA and the square of the distance between this area and that axis.
With respect to the central axis $G\overline{y}$ and $G\overline{z}$ (Fig.2.6) the moments of inertia are:

$$\boxed{I_{\bar{y}} = \int_A \overline{z}^2 dA} \quad (2.30)$$

and

$$\boxed{I_{\bar{z}} = \int_A \overline{y}^2 dA} \quad (2.31)$$

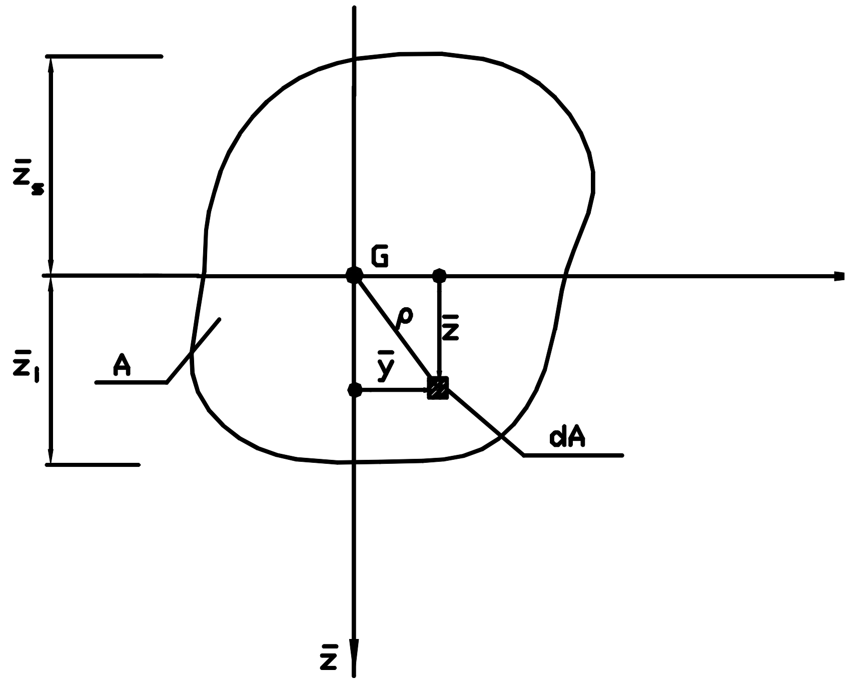


Fig.2.6

! The axial moments of inertia $I_{\bar{y}}$ and $I_{\bar{z}}$ are always *positive* and they are measured in [length]^4 units: cm^4 , m^4 , mm^4.

b. ***The centrifugal moment of inertia (the product moment of area)*** of an area *A* (Fig.2.6) with respect to a system of axis comprised in its plane represents the infinite sum of all products between the elementary area *dA* and the distances between this area and that rectangular system of axis.

With respect to the central system of axis $\bar{y}\,G\,\bar{z}$ (Fig.2.6) the centrifugal moment of inertia is:

$$\boxed{I_{\bar{y}\bar{z}} = \int_A \bar{y}\bar{z}\,dA} \qquad (2.32)$$

The centrifugal moment of inertia reflects through *value* and *signs* the cross section repartition in the fourth quadrants of the system of axis (Fig.2.6). It is also measured in [length]^4 units: cm^4 , m^4 , mm^4.

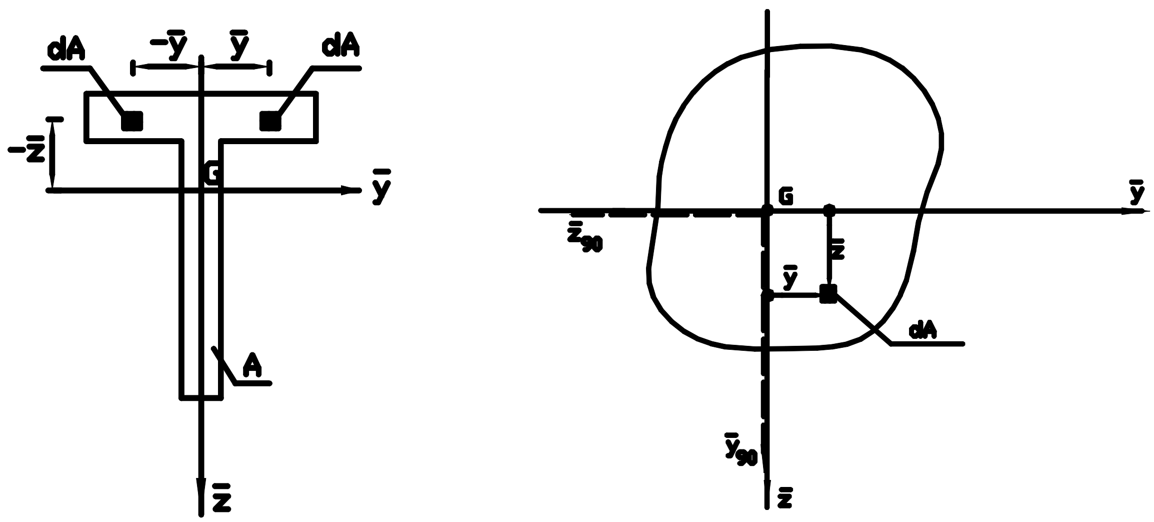
! While $I_{\bar{y}}$ and $I_{\bar{z}}$ are always *positive*, the centrifugal moment of inertia $I_{\bar{y}\bar{z}}$ may be *positive*, *negative* or even *zero.*

Proprieties of the centrifugal moment of inertia:

- If the cross section has at least an axis of symmetry (Fig.2.7.a), the centrifugal moment of inertia $I_{\bar{y}\bar{z}}$ is equal to zero.

Assuming the cross section from figure 2.7.a can be described as a pair of elementary areas, symmetrically disposed, the centrifugal moment of inertia $I_{\bar{y}\bar{z}}$ is:

$$I_{\bar{y}\bar{z}} = \int_A \overline{y}\overline{z}\,dA = \int_A \left[\overline{y}(-z)dA + (-y)(-z)dA\right] = 0$$



a. b.

Fig.2.7

- If the system of axis $\overline{y}\,\mathrm{G}\,\overline{z}$ is rotated about G with 90^0 in the clock-wise direction (Fig.2.7.b) the centrifugal moment of inertia changes its sign.

From figure 2.7.b we may write the relations between the initial system $\overline{y}\,\mathrm{G}\,\overline{z}$ and the rotated one $\overline{y}_{90}\,\mathrm{G}\,\overline{z}_{90}$:

$$\overline{y}_{90} = \overline{z}\ ;\ \overline{z}_{90} = \overline{y}$$

Replacing in the centrifugal moment of inertia:

$$I_{\bar{y}\bar{z}} = \int_A \overline{y}\overline{z}\,dA = \int_A \left(-\overline{z_{90}}\right)\overline{y_{90}}\,dA = -I_{\overline{y_{90}}\,\overline{z_{90}}}$$

c. *The polar moment of inertia*

It measure also the cross section inertia, but the rotation is made around an axis perpendicular to the cross section plane, which intersect the cross section in the centroid G.

From figure 2.6 we may write:

$$\rho^2 = \overline{z}^2 + \overline{y}^2 \qquad (2.33)$$

The polar moment of inertia, written similar to the axial moment of inertia, will be with (2.33):

$$I_p = \int_A \rho^2 dA = \int_A \left(\overline{z^2} + \overline{y^2}\right) dA = \int_A \overline{z}^2 dA + \int_A \overline{y}^2 dA$$

Result:

$$\boxed{I_p = I_{\bar{y}} + I_{\bar{z}}} \qquad (2.34)$$

From (2.34) we may conclude that the sum of the axial moments of inertia with respect to rectangular axis with the same origin in G (Fig.2.6) is an invariant to the rotation of the system of axis.

d. ***The radius of gyration (radius of inertia)***

If we want to increase the moment of inertia of a cross section with respect to an axis, then we move it away from this, without increasing the cross section area.

As the material consumption is directly proportional to the cross section area, the above solution is as economic as the ratio *I/A* is bigger.

The characteristic connected to the moment of inertia of a cross section with respect to an axis is *the radius of gyration*.

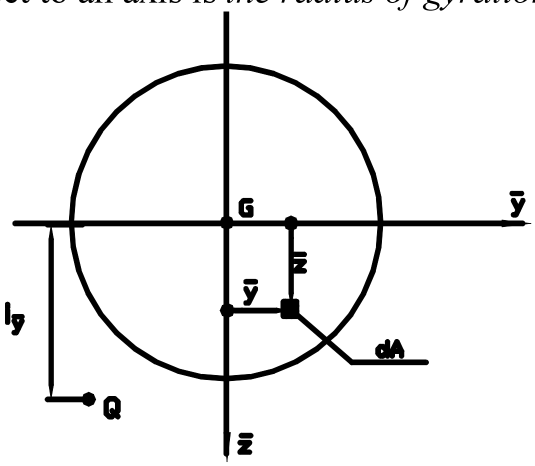


Fig.2.8

It is defined (Fig.2.8) as *the distance from an inertia axis to a fictitious point Q, in which if the entire cross section area A is concentrated, the moment of inertia (punctual) with respect to that axis is equal to the real moment of inertia.*

According to this definition we may write the equality:

$$I_{\bar{y}} = \int_A \bar{z}^2 dA = i_{\bar{y}}^2 \cdot A \tag{2.35}$$

From (2.35) the radius of gyration with respect to $G\bar{y}$ axis is:

$$\boxed{i_{\bar{y}} = \sqrt{\frac{I_{\bar{y}}}{A}}} \tag{2.36}$$

Similarly with respect to the other axis $G\bar{z}$:

$$\boxed{i_{\bar{z}} = \sqrt{\frac{I_{\bar{z}}}{A}}} \tag{2.37}$$

The radius of gyration is always measured in [length] units: cm, m, mm.

e. ***The strength modulus***

The ratio between the axial moment of inertia and the distance to the farthest point of the cross section from that axis is called *strength modulus.*

In figure 2.6 we note the distances from $G\bar{y}$ axis till the extreme points: superior $\bar{z}_s$ and inferior $\bar{z}_i$, and the strength modulus with respect to $G\,\bar{y}$ axis in these points are:

$$\boxed{W_{\overline{y}s} = \frac{I_{\overline{y}}}{z_s}} \quad \text{and} \quad \boxed{W_{\overline{y}i} = \frac{I_{\overline{y}}}{z_i}} \qquad (2.38)$$

The strength modulus can be positive or negative and it is always measured in $[\text{length}]^3$ units: cm^3 , m^3 , mm^3.

2.2.3 Moments of inertia for some simple cross sections

a. ***Rectangle***

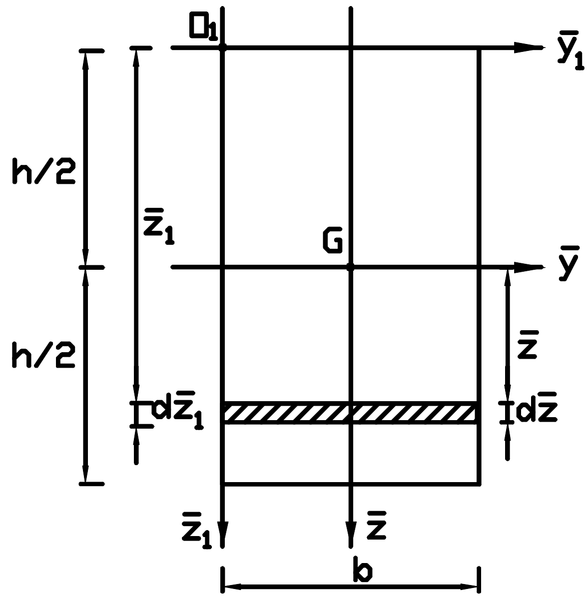


Fig.2.9

We consider the rectangular section from figure 2.9, for which we want to calculate through direct integration the moments of inertia with respect to $O\overline{y}_1$ and $O\overline{z}_1$ axis, tangent to the rectangle sides, and then with respect to the central axis $G\overline{y}$ and $G\overline{z}$.

We consider an elementary area dA, a rectangular strip parallel to $G\overline{y}$ axis, of width b and height $d\overline{z}$, so that $dA = b \cdot d\overline{z}$.

With respect to $O_1\overline{y}_1$ and $O_1\overline{z}_1$, $dA = b \cdot d\overline{z_1}$ (Fig.2.9) and the area, the static moments and the centroid coordinates will be:

$$A = \int_A dA = \int_0^h b \cdot d\overline{z_1} = bh$$

$$S_{\overline{y_1}} = \int_A \overline{z_1} dA = b\int_0^h \overline{z_1} \cdot d\overline{z_1} = \frac{bh^2}{2}$$

$$S_{\overline{z_1}} = \int_A \frac{b}{2} dA = \frac{b^2}{2}\int_0^h d\overline{z_1} = \frac{b^2 h}{2}$$

$$\overline{y_G} = \frac{S_{\overline{z_1}}}{A} = \frac{b}{2} \quad \text{and} \quad \overline{z_G} = \frac{S_{\overline{y_1}}}{A} = \frac{h}{2}$$

The axial moments of inertia with respect to $O_1\overline{y}_1$ and $O_1\overline{z}_1$ are:

$$I_{\overline{y_1}} = \int_A \overline{z_1}^2 dA = b\int_0^h \overline{z_1}^2 d\overline{z_1} = \frac{bh^3}{3}$$

and respectively:

$$I_{\overline{z_1}} = \frac{hb^3}{3}$$

The centrifugal moment of inertia:

$$I_{\overline{y_1}\overline{z_1}} = \int_A \overline{y_1}\,\overline{z_1} dA = \int_0^h \frac{b}{2}\overline{z_1} \cdot b \cdot d\overline{z_1} = \frac{b^2h^2}{4}$$

With respect to the central system of axis $\overline{y}$ **G** $\overline{z}$, $dA = b \cdot dz_1$ (Fig.2.9) and the same characteristics are now:

$$A = \int_A dA = \int_{-h/2}^{h/2} b \cdot d\overline{z} = b\left[\frac{h}{2} - \left(-\frac{h}{2}\right)\right] = bh$$

$$S_{\overline{y}} = \int_A \overline{z} dA = b \int_{-h/2}^{h/2} \overline{z} \cdot d\overline{z} = b\left[\frac{\left(\frac{h}{2}\right)^2}{2} - \frac{\left(-\frac{h}{2}\right)^2}{2}\right] = 0$$

Similarly:

$$S_{\overline{z}} = 0$$

We found a result that was obvious, because G$\overline{y}$ and G$\overline{z}$ are even the central axis, and we have seen in paragraph 2.2.1 that with respect to these axis $S_{\overline{y}} = S_{\overline{z}} = 0$.

Implicit:

$$\overline{y_G} = \overline{z_G} = 0$$

The axial moments of inertia with respect to $G\overline{y}$ and $G\overline{z}$ are:

$$I_{\overline{y}} = \int_A \overline{z^2} dA = b \int_{-h/2}^{h/2} \overline{z^2} d\overline{z} = \frac{b}{3}\left[\frac{h^3}{8} - \left(-\frac{h^3}{8}\right)\right] = \frac{b}{3} \cdot \frac{2h^3}{8}$$

$$\boxed{I_{\overline{y}} = \frac{bh^3}{12}} \tag{2.39}$$

Similarly:

$$\boxed{I_{\overline{z}} = \frac{b^3h}{12}} \tag{2.40}$$

The centrifugal moment of inertia:

$$\boxed{I_{\overline{y}\overline{z}} = 0} \tag{2.41}$$

Relation (2.41) results from the first property of the centrifugal moment of inertia, as the rectangle has 2 symmetry axes.

b. ***Triangle***

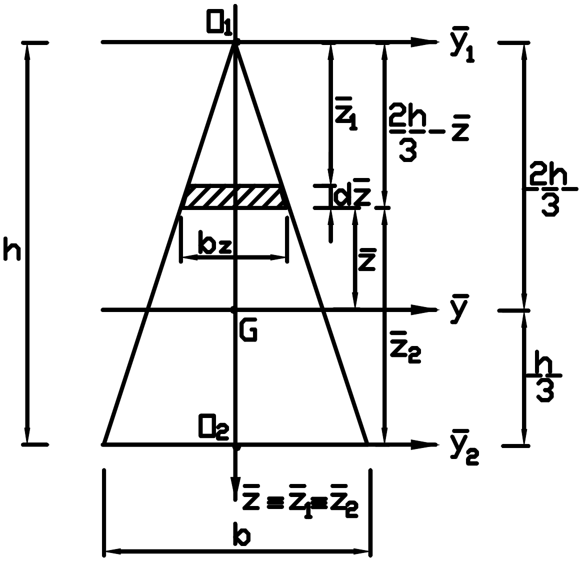


Fig.2.10

Let's consider the triangle from figure 2.10 of width b and height h. We want to calculate the moments of inertia with respect to $O_1\overline{y}_1$ and $O_1\overline{z}_1$ axes, with the origin O_1 in the triangle vertex, then with respect to $O_2\overline{y}_2$ and $O_2\overline{z}_2$ axes and finally with respect to the central axis of the triangle, $G\overline{y}$ and $G\overline{z}$.

We consider an elementary area dA as a rectangular strip parallel to $G\overline{y}$ axis, with the width b_z and height $d\overline{z}$, so that $dA = b_z \cdot d\overline{z}$.

For $I_{\overline{y}_1}$ we may write $b_z = b\dfrac{\overline{z_1}}{h}$ and $dA = b\dfrac{\overline{z_1}}{h}\cdot d\overline{z}$. With these:

$$I_{\overline{y}_1} = \int_A \overline{z_1}^2 dA = \int_0^h \overline{z_1}^2 \cdot b\frac{\overline{z_1}}{h}\cdot d\overline{z} = \frac{b}{h}\int_0^h \overline{z_1}^3 d\overline{z} = \frac{bh^3}{4}$$

For $I_{\overline{y}_2}$: $b_z = \dfrac{b}{h}\left(h - \overline{z_2}\right)$ and $dA = \dfrac{b}{h}\left(h-\overline{z_2}\right)\cdot d\overline{z}$. Therefore:

$$I_{\overline{y}_2} = \int_A \overline{z_2}^2 dA = \int_0^h \overline{z_2}^2 \cdot \frac{b}{h}\left(h-\overline{z_2}\right)\cdot d\overline{z} = \frac{bh^3}{12}$$

For $I_{\overline{y}}$: $b_z = \dfrac{b}{h}\left(\dfrac{2}{3}h - \overline{z}\right)$ and $dA = \dfrac{b}{h}\left(\dfrac{2}{3}h-\overline{z}\right)\cdot d\overline{z}$. Hence:

$$I_{\overline{y}} = \int_A \overline{z}^2 dA = \int_0^h \overline{z}^2 \cdot \frac{b}{h}\left(\frac{2}{3}h-\overline{z}\right)\cdot d\overline{z} = \frac{bh^3}{36}$$

Similarly we may write the moments of inertia $I_{\overline{z}_1}$, $I_{\overline{z}_2}$ and $I_{\overline{z}}$.

We may keep in mind the moments of inertia of the triangle, with respect to the central axes:

$$\boxed{I_{\bar{y}} = \frac{bh^3}{36}} \qquad \boxed{I_{\bar{z}} = \frac{b^3h}{36}} \qquad \boxed{I_{\bar{y}\bar{z}} = \frac{b^2h^2}{72}} \tag{2.42}$$

c. ***Circle***

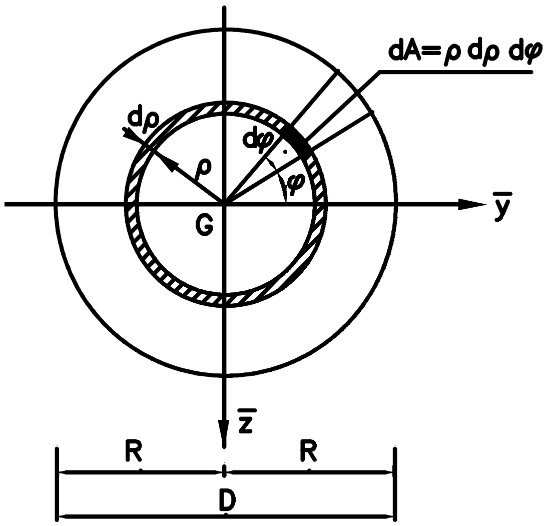


Fig.2.11

The elementary area dA (Fig.2.11) is comprised between two radius situated to an angle $d\varphi$ and two concentric circles of radius ρ and $\rho + d\rho$:

$dA = \rho \cdot d\rho \cdot d\varphi$

With the limits of integration:

$0 \le \rho \le R$

and

$0 \le \varphi \le 2\pi$

and integrating on the entire circle surface, we get:

$$A = \int_A dA = \int_0^R \rho \cdot d\rho \int_0^{2\pi} d\varphi = \frac{R^2}{2} \cdot 2\pi = \pi R^2 = \frac{\pi D^2}{4}$$

The distances from the elementary area centroid to $G\bar{z}$ axis, respectively $G\bar{y}$ axis are:

$$\bar{y} = \rho \cos\varphi$$
$$\bar{z} = -\rho \sin\varphi$$

The axial moments of inertia with respect to $G\bar{y}$ and $G\bar{z}$ are:

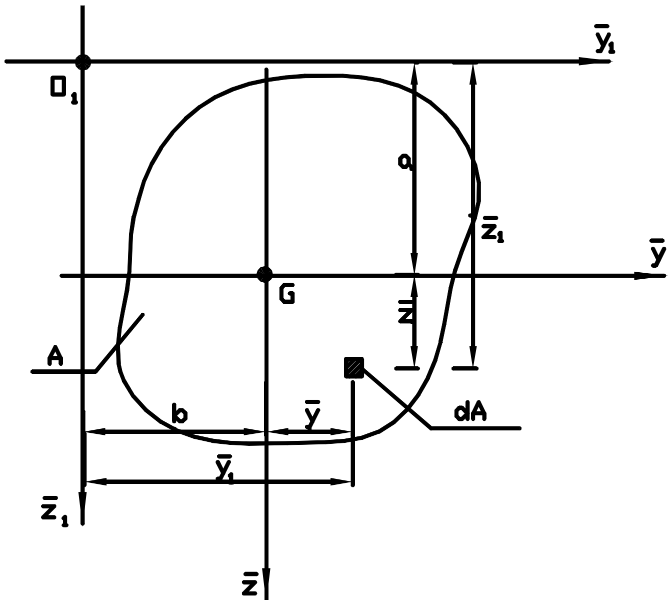
$$I_{\bar{y}} = \int_A \bar{z}^2 dA = \int_0^h \rho^2 \sin^2\varphi \cdot \rho \cdot d\rho \cdot d\varphi = \frac{R^4}{4} \int_0^{2\pi} \frac{1 - \cos 2\varphi}{2} d\varphi = \frac{R^4}{8} \left(\varphi - \frac{\sin 2\varphi}{2} \right) \Bigg|_0^{2\pi} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}$$

So, the moments of inertia of the circle, with respect to the central axes, are:

$$\boxed{I_{\bar{y}} = \frac{\pi D^4}{64} = I_{\bar{z}}} \quad \text{and} \quad \boxed{I_{\bar{y}\bar{z}} = 0} \tag{2.43}$$

2.2.4 The variation of the moments of inertia with the translation of axes

Let's consider a plane cross section of area A with an orthogonal system of axis $\bar{y}O\bar{z}$. Knowing the values of the moments of inertia $I_{\bar{y}}$ and $I_{\bar{z}}$, we want to determine the moments of inertia $I_{\bar{y}_1}$ and $I_{\bar{z}_1}$ with respect to $O_1\bar{y}_1$ and $O_1\bar{z}_1$ axis, parallel to the first axis (Fig.2.12).



$$\overline{y_1} = \overline{y} + b$$
$$\overline{z_1} = \overline{z} + a$$

Fig.2.12

With respect to $O_1\overline{y}_1$ axis, the moment of inertia is:

$$I_{\overline{y_1}} = \int_A \overline{z_1}^2 dA = \int_A \left(\overline{z} + a\right)^2 \cdot dA = \int_A \overline{z}^2 dA + 2\int_A \overline{z} a dA + \int_A a^2 dA$$

$$I_{\overline{y_1}} = I_{\overline{y}} + 2a \cdot S_{\overline{y}} + a^2 \cdot A$$

Similarly, with respect to $O_1\overline{z}_1$ axis, the moment of inertia is:

$$I_{\overline{z_1}} = I_{\overline{z}} + 2b \cdot S_{\overline{z}} + b^2 \cdot A$$

The centrifugal moment of inertia is:

$$I_{\overline{y_1}\overline{z_1}} = \int_A \left(\overline{z} + a\right)\left(\overline{y} + b\right) \cdot dA = \int_A \overline{y}\overline{z} \cdot dA + \int_A \overline{y} a \cdot dA + \int_A \overline{z} b \cdot dA + \int_A ab \cdot dA$$

$$I_{\overline{y_1}\overline{z_1}} = I_{\overline{y}\overline{z}} + a \cdot S_{\overline{z}} + b \cdot S_{\overline{y}} + ab \cdot A$$

In the above relations regarding $I_{\overline{y_1}}$, $I_{\overline{z_1}}$ and $I_{\overline{y_1}\overline{z_1}}$, $S_{\overline{y}}$ and $S_{\overline{z}}$ are static moments with respect to $O\overline{y}$ and $O\overline{z}$ axis. If these axes are even the central axes of the cross section ($O\overline{y} \equiv G\overline{y}$ and $O\overline{z} \equiv G\overline{z}$), these static moments are zero:

$$S_{\overline{y}} = S_{\overline{z}} = 0$$

The relations become:

$$\boxed{I_{\overline{y_1}} = I_{\overline{y}} + a^2 \cdot A} \qquad (2.44)$$

$$\boxed{I_{\overline{z_1}} = I_{\overline{z}} + b^2 \cdot A} \qquad (2.45)$$

$$\boxed{I_{\overline{y_1}\overline{z_1}} = I_{\overline{y}\overline{z}} + ab \cdot A} \qquad (2.46)$$

Relations (2.44), (2.45) and (2.46) represent the formulas of the moments of inertia with respect to translated axes, called *Steiner's formulas*. Each first term from relations represent the moment of inertia with respect to the centroid axes $G\overline{y}$ and

$G\bar{z}$, while the second term is the translation term equal with square distance between the translated axes multiplied by the corresponding area.

Jakob Steiner (1796-1863)

Inversely, if we have the moments of inertia with respect to a system of axes $\bar{y}_1\, O_1\, \bar{z}_1$, the moments of inertia with respect to the centroid axes $\bar{y}\, G\bar{z}$ are:

$$I_{\bar{y}} = I_{\bar{y}_1} - a^2 \cdot A$$
$$I_{\bar{z}} = I_{\bar{z}_1} - b^2 \cdot A$$
$$I_{\bar{y}\bar{z}} = I_{\bar{y}_1\bar{z}_1} - ab \cdot A$$

Let's check these formulas for the rectangular section. From 2.2.3.a paragraph, we got (Fig.2.9):

$$I_{\bar{y}_1} = \frac{bh^3}{3},\ I_{\bar{z}_1} = \frac{b^3h}{3},\ I_{\bar{y}_1\bar{z}_1} = \frac{b^2h^2}{4}$$

Replacing in the relation written above (see Fig.2.9):

$$I_{\bar{y}} = \frac{bh^3}{3} - \left(\frac{h}{2}\right)^2 \cdot bh = \frac{bh^3}{12}$$
$$I_{\bar{y}} = \frac{b^3h}{3} - \left(\frac{b}{2}\right)^2 \cdot bh = \frac{b^3h}{12}$$
$$I_{\bar{y}\bar{z}} = \frac{b^2h^2}{4} - \left(\frac{b}{2}\right)\left(\frac{h}{2}\right) \cdot bh = 0$$

We found the relations (2.39), (2.40) and (2.41).

2.2.5 The variation of the moments of inertia with the rotation of axes

Let's assume that for a cross section, with respect to the central system of axes $\overline{y}\,G\,\overline{z}$, the moments of inertia are known and we intent to compute the moments of inertia with respect to an orthogonal system of axes $\overline{y_\alpha}\,G\,\overline{z_\alpha}$ (Fig.2.13), rotated with an angle α (α > 0 for clockwise sense).

From Fig.2.13.b:

$$GC = \overline{y}\,,\ CD = \overline{z}\,,\ GB = \overline{y_\alpha}\,,\ DB = \overline{z_\alpha}$$

$$GA = \frac{\overline{y}}{\cos\alpha}\,,\ AC = \overline{y}tg\alpha\,,$$

$$AD = CD - AC = \overline{z} - \overline{y}tg\alpha$$

$$AB = AD\sin\alpha = (\overline{z} - \overline{y}tg\alpha)\sin\alpha$$

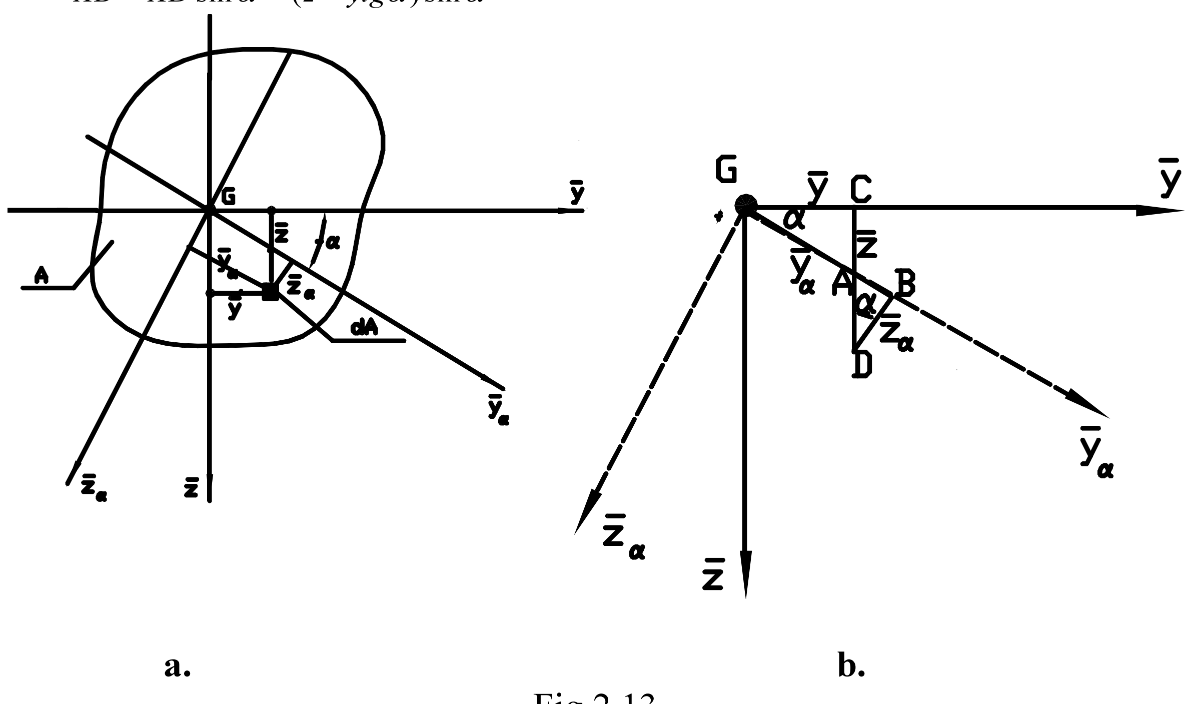


Fig.2.13

$$GB = GA + AB = \frac{\overline{y}}{\cos\alpha} + \overline{z}\sin\alpha - \overline{y}tg\alpha\sin\alpha = \frac{\overline{y}}{\cos\alpha}\left(1-\sin^2\alpha\right) + \overline{z}\sin\alpha = \frac{\overline{y}}{\cos\alpha}\cos^2\alpha + \overline{z}\sin\alpha$$

Finally the coordinates in the rotated system of axes, are:

$$\boxed{\overline{y_\alpha} = \overline{y}\cos\alpha + \overline{z}\sin\alpha} \tag{2.47}$$

$$DB = AD\cos\alpha = (\overline{z} - \overline{y}tg\alpha)\cos\alpha$$

$$\boxed{\overline{z_\alpha} = -\overline{y}\sin\alpha + \overline{z}\cos\alpha} \tag{2.48}$$

The moments of inertia with respect to the rotated system of axes $\overline{y_\alpha}\, G\, \overline{z_\alpha}$ are:

$$I_{\overline{y_\alpha}} = \int_A \overline{z_\alpha}^2 dA = \int_A \left(-\overline{y}\sin\alpha + \overline{z}\cos\alpha\right)^2 \cdot dA = \sin^2\alpha \int_A \overline{y}^2 dA - 2\sin\alpha\cos\alpha \int_A \overline{y}\,\overline{z} dA + \cos^2\alpha \int_A \overline{z}^2 dA$$

Replacing (2.30), (2.31) and (2.32) we have:

$$I_{\overline{y_\alpha}} = I_{\overline{y}}\cos^2\alpha + I_{\overline{z}}\sin^2\alpha - I_{\overline{y}\overline{z}}\sin 2\alpha$$

In a similar manner we find the other central moment of inertia and the centrifugal moment of inertia:

$$I_{\overline{z_\alpha}} = I_{\overline{y}}\sin^2\alpha + I_{\overline{z}}\cos^2\alpha + I_{\overline{y}\overline{z}}\sin 2\alpha$$

$$I_{\overline{y_\alpha}\overline{z_\alpha}} = \frac{I_{\overline{y}} - I_{\overline{z}}}{2}\sin 2\alpha + I_{\overline{y}\overline{z}}\cos 2\alpha$$

Replacing $\sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$ and $\cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}$, the moments of inertia are:

$$\boxed{I_{\overline{y_\alpha}} = \frac{I_{\overline{y}} + I_{\overline{z}}}{2} + \frac{I_{\overline{y}} - I_{\overline{z}}}{2}\cos 2\alpha - I_{\overline{y}\overline{z}}\sin 2\alpha} \qquad (2.49)$$

$$\boxed{I_{\overline{z_\alpha}} = \frac{I_{\overline{y}} + I_{\overline{z}}}{2} - \frac{I_{\overline{y}} - I_{\overline{z}}}{2}\cos 2\alpha + I_{\overline{y}\overline{z}}\sin 2\alpha} \qquad (2.50)$$

$$\boxed{I_{\overline{y_\alpha}\overline{z_\alpha}} = \frac{I_{\overline{y}} - I_{\overline{z}}}{2}\sin 2\alpha + I_{\overline{y}\overline{z}}\cos 2\alpha} \qquad (2.51)$$

Adding the first two relations we get:

$$I_{\overline{y_\alpha}} + I_{\overline{z_\alpha}} = I_{\overline{y}} + I_{\overline{z}}$$

In conclusion, the sum of the moments of inertia with respect to orthogonal axes having the same origin is an invariant.

2.2.6 Principal moments of inertia. Principal axis of inertia

The moments of inertia $I_{\overline{y_\alpha}}$ and $I_{\overline{z_\alpha}}$ are continuous and periodical functions of *α*. For these functions, we can find a value of *α* which correspond to an extreme value of the moments of inertia. We make the first derivative of $I_{\overline{y_\alpha}}$ and we equalize it 0:

$$\frac{dI_{\overline{y_\alpha}}}{d\alpha} = -(I_{\overline{y}} - I_{\overline{z}})\sin 2\alpha - 2I_{\overline{y}\overline{z}}\cos 2\alpha = 0$$

We get, from this equation:

$$\boxed{tg 2\alpha = -\frac{2I_{\overline{y}\overline{z}}}{I_{\overline{y}} - I_{\overline{z}}}} \qquad (2.52)$$

But also:

$$\frac{dI_{\overline{y_\alpha}}}{d\alpha} = -2\left(\frac{I_{\overline{y}} - I_{\overline{z}}}{2}\sin 2\alpha + I_{\overline{y}\overline{z}}\cos 2\alpha\right) = -2I_{\overline{y_\alpha}\overline{z_\alpha}} = 0$$

We get that $I_{\overline{y_\alpha}\,\overline{z_\alpha}} = 0$, what means that axes $\overline{y_\alpha}$ and $\overline{z_\alpha}$ are conjugated axes, so with respect to these axes the central moments of inertia $I_{\overline{y}}$ and $I_{\overline{z}}$ have extreme values. We shall name these axes *principal axes of inertia* (noted with *Gy* and *Gz*) and the corresponding moments of inertia are *the principal moments of inertia* (noted with I_y and I_z).

The principal axes orientation is given by the equation: $tg2\alpha = -\frac{2I_{\overline{y}\overline{z}}}{I_{\overline{y}} - I_{\overline{z}}}$, which has the solutions: $2\beta_i = 2\alpha_0 + n\pi \quad (n = 0,1,2,.....)$

The first two solutions are:

$$n = 0 \Rightarrow 2\beta_1 = 2\alpha_0 \Rightarrow \beta_1 = \alpha_0$$

$$n = 1 \Rightarrow 2\beta_2 = 2\alpha_0 + \pi \Rightarrow \beta_2 = \alpha_0 + \frac{\pi}{2}$$

where:

$$2\alpha_0 = arctg\left(-\frac{2I_{\overline{y}\overline{z}}}{I_{\overline{y}} - I_{\overline{z}}}\right)$$

In conclusion, the principal axes of inertia are orthogonal axes.

We agree that $|\beta_1| \le \frac{\pi}{2}$

We write another condition of extreme, for $\alpha=\beta_1$:

$$\left(\frac{d^2 I_{\overline{y}\alpha}}{d\alpha^2}\right)_{\alpha=\beta_1} < 0 :$$

$$-(I_{\overline{y}} - I_{\overline{z}})2cos2\beta_1 + 4I_{\overline{yz}}sin2\beta_1 < 0$$

But: $I_{\overline{y}} - I_{\overline{z}} = -\frac{2I_{\overline{yz}}}{tg2\beta_1}$

$$\rightarrow \frac{2I_{\overline{yz}}}{tg2\beta_1}2cos2\beta_1 + 4I_{\overline{yz}}sin2\beta_1 < 0$$

$$\rightarrow \frac{4I_{\overline{yz}}}{sin2\beta_1}(cos^2 2\beta_1 + sin^2 2\beta_1) < 0 \rightarrow \frac{4I_{\overline{yz}}}{sin2\beta_1} < 0$$

This final relation tells us that the centrifugal moment of inertia $I_{\overline{yz}}$ and the angle β_1 must always have different signs. Always *Gy* is the strong axis of inertia, so:

$$I_y > I_z$$

With relations (2.49), (2.50) and (2.52) the final relations for *the principal moments of inertia* I_y and I_z are:

$$\boxed{I_{y,z} = \frac{I_{\overline{y}}+I_{\overline{z}}}{2} \pm \frac{1}{2}\sqrt{\left(I_{\overline{y}} - I_{\overline{z}}\right)^2 + 4I_{\overline{yz}}^2}} \tag{2.53}$$

The principal axes of inertia are those to which the centrifugal moment of inertia is zero.

If a cross section has at least one axis of symmetry, the central axes of inertia are identically to the principal axes of inertia.

Application

For the quarter-circle in the figure bellow (Fig.2.14) calculate the principal moments of inertia and the direction of the principal axes.

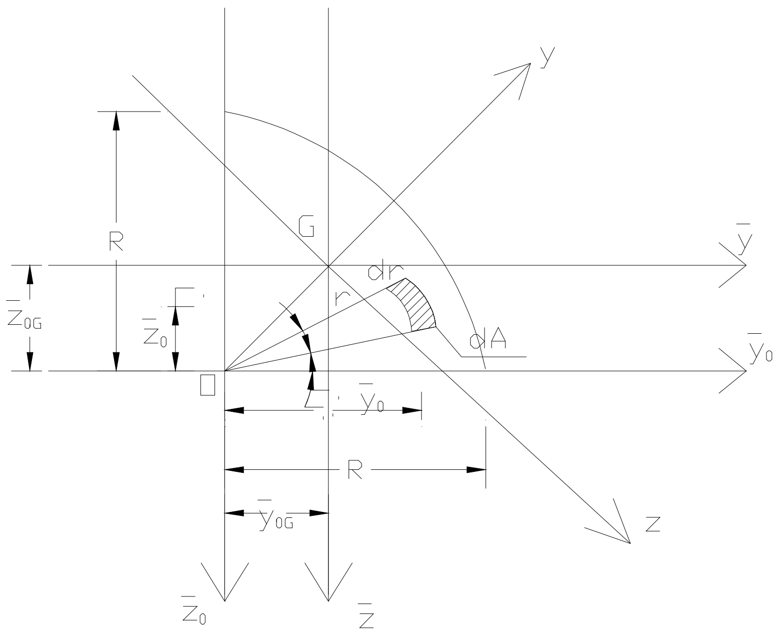


Fig.2.14

In the chosen reference system $\overline{y}_0O\overline{z}_0$ the differential element of area *dA* written in polar coordinates is:

$$dA = r \cdot d\varphi \cdot dr$$

The area A is:

$$A = \int_A dA = \int_A r \cdot d\varphi \cdot dr = \int_0^R r \cdot dr \int_0^{\pi/2} d\varphi = \frac{\pi R^2}{4}$$

The distances from the centroid of the differential element to the axes $O\overline{z}_0$, respectively $O\overline{y}_0$ are:

$$\overline{y}_0 = r \cdot cos\varphi \qquad \overline{z}_0 = -r \cdot sin\varphi$$

The second moments of area (static moments) about $O\overline{y}_0$ axis, respectively $O\overline{y}_0$ axis, are:

$$S_{\overline{y}_0} = \int_A \overline{z}_0 dA = -\int_A r \cdot sin\varphi \cdot r \cdot d\varphi \cdot dr = -\int_0^R r^2 dr \int_0^{\pi/2} sin\varphi d\varphi = -\frac{R^3}{3}$$

$$S_{\overline{z}_0} = \int_A \overline{y}_0 dA = \int_A r \cdot cos\varphi \cdot r \cdot d\varphi \cdot dr = \int_0^R r^2 dr \int_0^{\pi/2} cos\varphi d\varphi = \frac{R^3}{3}$$

The coordinates of the centroid:

$$\overline{y}_{0G} = \frac{S_{\overline{z}_0}}{A} = \frac{\frac{R^3}{3}}{\frac{\pi R^2}{4}} = \frac{4R}{3\pi}$$

$$\overline{z}_{0G} = \frac{S_{\overline{y}_0}}{A} = \frac{-\frac{R^3}{3}}{\frac{\pi R^2}{4}} = -\frac{4R}{3\pi}$$

In order to calculate the central moments of inertia $I_{\overline{y}}$ and $I_{\overline{z}}$ and the centrifugal moment of inertia $I_{\overline{yz}}$ we calculate first the moments of inertia about the reference axes, $I_{\overline{y}_0}$, $I_{\overline{z}_0}$and $I_{\overline{y}_0\overline{z}_0}$:

$$I_{\overline{y}_0} = \int_A \overline{z_0}^2 dA = \int_A r^2 \cdot sin^2\varphi \cdot r \cdot d\varphi \cdot dr = \int_0^R r^3 dr \int_0^{\pi/2} sin^2\varphi d\varphi = \frac{\pi R^4}{16}$$

$$I_{\overline{z}_0} = \int_A \overline{y_0}^2 dA = \int_A r^2 \cdot cos^2\varphi \cdot r \cdot d\varphi \cdot dr = \int_0^R r^3 dr \int_0^{\pi/2} cos^2\varphi d\varphi = \frac{\pi R^4}{16}$$

$$I_{\overline{y}_0\overline{z}_0} = \int_A \overline{y}_0\overline{z}_0 dA = -\int_A rsin\varphi \cdot rcos\varphi \cdot r \cdot d\varphi \cdot dr = -\int_0^R r^3 dr \int_0^{\pi/2} sin\varphi \cdot cos\varphi d\varphi = -\frac{R^4}{8}$$

The central moments of inertia $I_{\overline{y}}$ and $I_{\overline{z}}$ are calculated by translating:

$$I_{\overline{y}} = I_{\overline{y}_0} - \overline{z}_{0G}{}^2 A = \frac{\pi R^4}{16} - \left(-\frac{4R}{3\pi}\right)^2 \frac{\pi R^2}{4} = R^4\left(\frac{\pi}{16} - \frac{4}{9\pi}\right)$$

$$I_{\overline{z}} = I_{\overline{z}_0} - \overline{y}_{0G}{}^2 A = \frac{\pi R^4}{16} - \left(\frac{4R}{3\pi}\right)^2 \frac{\pi R^2}{4} = R^4\left(\frac{\pi}{16} - \frac{4}{9\pi}\right)$$

$$I_{\overline{yz}} = I_{\overline{y}_0\overline{z}_0} - \overline{y}_{0G}\overline{z}_{0G} \cdot \quad A = -\frac{R^4}{8} - \left(\frac{4R}{3\pi}\right)^2 \frac{\pi R^2}{4} = -R^4\left(\frac{1}{8} + \frac{4}{9\pi}\right)$$

The principal moments of inertia calculated with formula (2.53) are:

$I_y = R^4\left(\frac{\pi}{16} + \frac{1}{8}\right)$ and $I_z = R^4\left(\frac{\pi}{16} - \frac{1}{8} - \frac{8}{9\pi}\right)$

The rotation of the principal axes: $tg\ 2\alpha = -\frac{2I_{\overline{yz}}}{I_{\overline{y}}-I_{\overline{z}}} = \infty$

The solutions are: $2\alpha = \frac{\pi}{2}$ and $2\alpha - \pi = -\frac{\pi}{2}$

The first solution is the correct one, due to the condition: $\frac{tg2\alpha}{I_{\overline{yz}}} < 0$

The final angle of rotation: $\beta = \frac{\pi}{4}$