Buletinul Ştiinţific al Universităţii "POLITEHNICA" din Timişoara

Seria HIDROTEHNICA

TRANSACTIONS on HYDROTECHNICS

Tom 56(70), Fascicola 2, 2011

Criteria for stability in numerical methods

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Abstract: **Numerical solutions of convection-dispersion equation are numerous, but Finite Difference method has disadvantages two undesirable characteristics: stability criterion and numerical dispersion.**

Like most explicit schemes, this method is only under certain conditions. If the stability criterion is not met, the numerical model is prone to oscillations in space or time.

The paper makes a comparative analysis between explicit forms central and upwind and analytical method of a homogeneous and isotropic system, 1D. Results will emphasize the upwind form is more stable than the central form, regardless of variation of Peclet

Keywords: convection-dispersion equation, Numerical solutions, Finite Difference method, oscillations, stability criterion, Peclet number

1. INTRODUCTION

Once the conceptual model is translated into the mathematical model associated with the initial conditions and to limit, a solution can be obtained by transforming it into a numerical model and using a software solution.

Mathematical models in hydrogeology are based on the continuum, the real system is considered continuous in space and time. Using continuous variables, the mathematical model is expressed analytically, for homogeneous and isotropic case.

Analytical models and corresponding solutions characterize small areas, allowing significant simplifications of real systems. Groundwater problems can be solved analytically so, no-errors (Spitz and Moreno, 1996).

Analytical models cannot be used when the parameters vary in the amount studied or when conditions are complex edge. In transport problems due to their complexity, increasing the size problem, analytical solutions are valid for 1D case. If the mathematical model cannot be solved analytically be transformed into a numerical model and then solved.

Numerical methods allowing the solution to a restricted set of points distributed convenient grid using Cartesian coordinate system. Partial differential equations are solved by discrete schemes, concentrations are constant in an element, but vary between different elements. In this way, a heterogeneous aquifer is approximated as a collection of different homogeneous regions. Solutions are obtained in network nodes.

2. STABILITY ANALYSIS¶

It will take into account the 1D case, the convection - dispersion with the following expression:

$$\frac{\partial c}{\partial t} + v_a \frac{\partial c}{\partial x} - D\frac{\partial^2 c}{\partial x^2} = 0 \qquad (1)$$

Transport equation is imposed initial and boundary conditions:

$$c(x=0, t)=c_0 f(t) \qquad (2)$$

$$\frac{\partial^2 c}{\partial x^2} = 0, for\ x = L \qquad (3)$$

Discretisation in space and time of a field with three aspects of finite differences is shown in fig1.

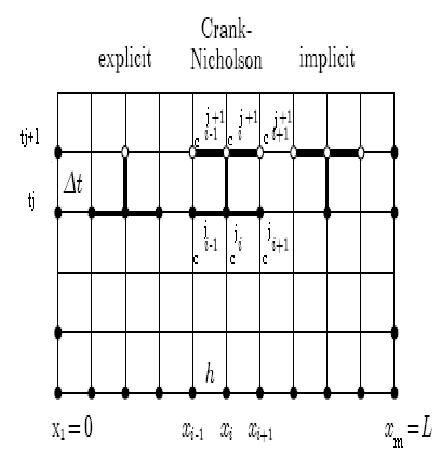

Fig. 1 An aquifer in cell division depending on space and time

The two forms of upwind calculation are:

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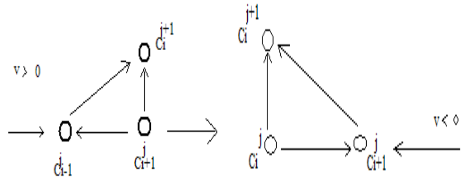

Fig.2 Forms account for upwind

Calculation of finite differences method is:

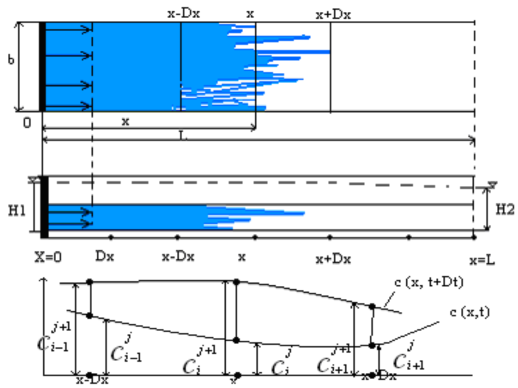

Fig.3 Calculation of finite differences method

Depending on the direction of the mesh (spatial and temporal) equations are transformed into Taylor series:

- for x constant

$$- \quad c(x,t+\Delta t) = c(x,t) + \Delta t \frac{\partial c}{\partial t_{x,t}} + \frac{\Delta t^2 \partial^2 c}{2\partial t^2} + \frac{\Delta t^3}{3!}\frac{\partial^3 c}{\partial t^3} + .. \qquad (4)$$

$$- \quad c(x,t) = c(x,t+\Delta t) - \Delta t \frac{\partial c}{\partial t_{x,t}} + \frac{\Delta t^2 \partial^2 c}{2\partial t^2} - \frac{\Delta t^3}{3!}\frac{\partial^3 c}{\partial t^3} + ... \qquad (5)$$

- for t constant

$$c(x+\Delta x,t) = c(x) + \Delta x \frac{\partial c}{\partial x_{x,t}} + \frac{\Delta x^2}{2}\frac{\partial^2 c}{\partial x^2} + \frac{\Delta x^3}{3!}\frac{\partial^3 c}{\partial x^3} + ... \qquad (6)$$

$$c(x-\Delta x,t) = c(x) - \Delta x \frac{\partial c}{\partial x_{x,t}} + \qquad (7)$$

$$\frac{\Delta x^2}{2}\frac{\partial^2 c}{\partial x^2} - \frac{\Delta x^3}{3!}\frac{\partial^3 c}{\partial x^3} + ... \qquad (8)$$

For i=0,1,2,....m and j=0.1,2,3,...n

Explicit form has the following expression:

$$c_i^{j+1} = c_i^j +$$

$$\frac{D\Delta t}{\Delta x^2}\{(1+\gamma Pe)c_{i-1}^j - [2-(1-2\gamma)Pe+Ze]c_i^{j+1} + [1-(1-\gamma)Pe]c_{i+1}^{j+1}\} \qquad (9)$$

$$c_i^{j+1} = c_i^j + \gamma Cr c_{i-1}^j + (1-2\gamma)Cr c_i^j - (1-\gamma)Cr c_{i+1}^j \qquad (10)$$

Where:

$$Pe = \frac{v_a \Delta x}{D} \text{ (David, 98, pg167)} \qquad (11)$$

$$Ze = \frac{\lambda \Delta x^2}{D} \text{ is the number of Zerfall} \qquad (12)$$

$$Cr = \frac{v_a \Delta t}{\Delta x} \text{ Courant criterion} \qquad (13)$$

3. CASE STUDY

One way of solving the transport equation is the finite difference method is ASMWIN program 6.0, uses two types central and upwind.

Modeled area is shown in Fig.4

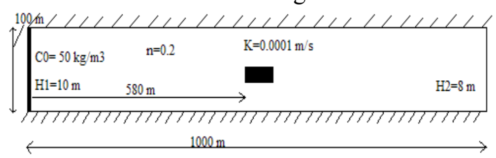

Fig.4 Modeled area

For the simplest case: if a D homogeneous environment, the relationship between analytical and finite differences whore two forms of the following issue:

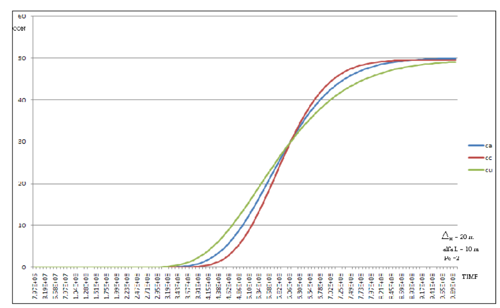
Fig.5 Correlation between analytical (as), the central finite differences (DC) and upwind finite differences (with) for convergence and stability field, on = 2, continuous injection

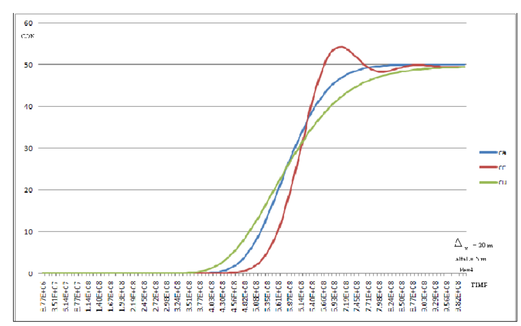
Fig. 6 Correlation between analytical (as), the central finite differences (DC) and upwind finite differences (with), the unstable area, on = 4, constant injection

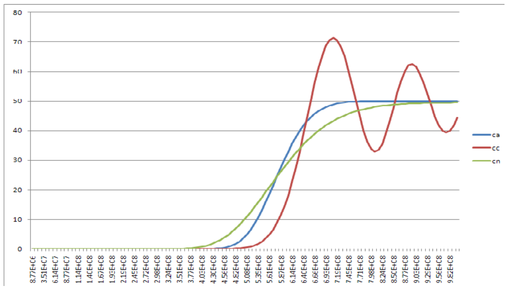

Fig. 7 Correlation between analytical (as), the central finite differences (DC) and upwind finite differences (with), the unstable area, On = 6.66, constant injection

As we see in graphic is more stable upwind form than central

Next I followed if the condition that the error at time t +1 is smaller than the error at time t.

Have been elected three times (T1, T2, T3) and followed the evolution of the error.

Table no.1.

$\alpha_L =$ 10 m	Time	$\xi = \frac{\lvert ca-cc \rvert}{ca}100$ or $\xi = \frac{\lvert ca-cu \rvert}{ca}100$	Analitic concentration	Central concentration form	Upwind concentration form
	T1=2.54E+08s	$\xi_{ca/cc}$ =81.33% si $\xi_{ca/cu}$ =83.13%	1.4E-4	7.51E-4	8.30E-4
	T2=5.17E+08s	$\xi_{ca/cc}$ = 34.42% $\xi_{ca/cu}$ =18.13%	13.59	10.11	16.60
	T3=8.68E+08s	$\xi_{ca/cc}$ =0.42% $\xi_{ca/cu}$ =3.33%	49.20	49.41	47.61
$\alpha_L = 5$ m					
	T1=2.54E+08s	$\xi_{ca/cc}$ =0 $\xi_{ca/cu}$ =0	3.19E-13	0	0
	T2=5.17E+08s	$\xi_{ca/cc}$ =259.39% $\xi_{ca/cu}$ = 31,17%	9.56	2.66	13.89
	T3=8.68E+08s	$\xi_{ca/cc}$ =0.38% $\xi_{ca/cu}$ =2.73%	49.94	49.75	48.61
$\alpha_L = 3$ m					
	T1=2.54E+08s	$\xi_{ca/cc}$ = 24.25% $\xi_{ca/cu}$ =99.99	2.78E-15	3.67E-15	5.63E-8
	T2=5.17E+08s	$\xi_{ca/cc}$ =742.85% $\xi_{ca/cu}$ =48.08	6.49	0.77	12.50
	T3=8.68E+08s	$\xi_{ca/cc}$ =11.98 % $\xi_{ca/cu}$ =2.08%	49.99	56.80	48.97

Past errors with red in table shows oscillations appeared to central form

4. CONCLUSIONS

The analysis of data obtained by running the program 6.0 ASMWIN and compare them with analytics observed upwind shape is stable and convergent, independent of the variation coefficient dispersivitate lengthwise and core form is unstable. In the criterion for compliance, the higher volatility increases, this condition is no longer respected. A great influence on the results of an assessment is the coefficient of dispersion, it's best for him to determine the ground.

5. BIBLIOGRAPHY

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