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Mathematical modelling of groundwater flow in shallow aquifer containing a cavity of arbitrary form bounded by a partially permeable contour

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Abstract: Groundwater flow in shallow aquifer (i.e. groundwater reservoir with large plane extension in relation to the depth) containing lake, pond, groundwater recharge or drainage pit, foundation pit etc., referred further as groundwater extraction/recharge cavity or cavern represent a very important practice-oriented topics.

In this regard in a former paper a general mathematical representation of groundwater flow in shallow aquifer is deduced, considering a cavity of arbitrary form bounded by a permeable contour, using the theory of the analytical functions of a complex variable.

In the present paper an extension of this problem will be presented, considering a cavity of arbitrary form bounded by a partially permeable contour. This extension allows approach of new aspects and issues of groundwater management. The mathematical representations consider asymptotic conditions determined by a pre-existing initial uniform groundwater flow which has an important influence on the flow processes especially in neighbourhood of cavity.

It will be deduced formulas which allow a rapid analysis of the groundwater balance in the modelled region considering the dependence of the recharge/discharge rate of the cavity from and the extension of the impermeable part of the contour and from the pre-existing uniform groundwater flow.

The obtained mathematical representations and formulas can be applied for cavities of different shape using conformal mapping defined through analytical functions of a complex variable.

1. INTRODUCTION

The main purpose in this paper is to present a mathematical method for two-dimensional groundwater flow in shallow aquifer (i.e.large plane extension in relation to the depth) containing a cavity of arbitrary shape bounded by a closed contour C which consist of a permeable portion C_0 and an impermeable portion C_Σ (defined through the points A_1 and B_1 Fig.1). In the cavity (i.e. inside of the contour C) there are free water table and a discharge/recharge (i.e. extraction/infiltration) rate of Q will be extracted or supplied. The cavity can be referred/represent in practical view a groundwater recharge/discharge system (e.g. groundwater recharge trench/pit, ecologic lake, pond, foundation pit, well with laterals and so on). The undisturbed flow state in aquifer a uniform groundwater flow, having a velocity of v_0, will be considered ($v_0 = 0$ correspond to an motionless groundwater basin).

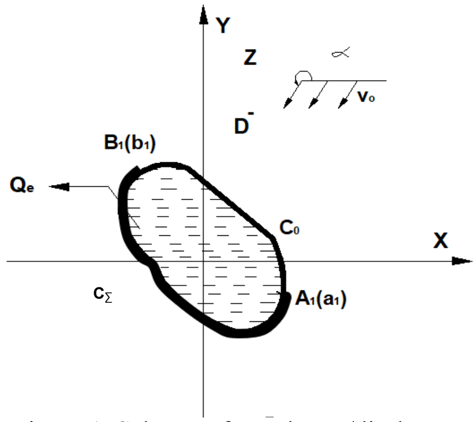

Figure 1. Scheme of a recharge/discharge cavity in shallow aquifer (plan view)

In a former paper [1] a general mathematical representation of groundwater flow in shallow aquifer containing a cavity of arbitrary form, bounded by a fully permeable contour (i.e. the impermeable portion C_Σ of the contour is missing) was discussed. Several other mathematical representation of groundwater flow one can see also in [3], [4],[5], [6], [7].

It is to mention also, that in opposite to the numerical methods the analytical solutions are expressed with mathematical functions (e.g. formulas) between of parameters that describe the modelled processes and, as a result, allow a faster and more efficient analysis regarding the influence of the different parameters. In case of groundwater flow modelling there are very performant numerical programs based on FDM [8] or FEM [9]. But, in special cases when the flow has singularities (e.g. well or endpoint of drainage trench) the numerical methods ca leaves to significant errors [6], [7]. That means that analytical methods stay up to date.

In the present paper an extension of this problem will be presented, considering also a cavity of arbitrary form which is bounded by a partially permeable contour (Fig.1), using the same theoretical basics (i.e. analytical functions of a complex variable) and the validity of Darcy's law.

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2. MATHEMATICAL MODELLING OF GROUNDWATER FLOW BY MEANS OF ANALYTICAL FUNCTIONS OF A COMPLEX VARIABLE

The basic equations of groundwater flow can be expressed using the scalar potential function Φ(x,y) of the velocity or the complex potential F(z) and the complex velocity W(z) both analytically functions of a complex variable (z=x+iy) [1],[2],[4].

The Darcy law for the flow velocity

$$\vec{v} = \text{grad}\Phi(x,y) = \nabla\Phi; \quad x,y \in D^- \qquad (1)$$

where Φ is the scalar potential function.

$$\Phi(x,y) = -k \cdot h(x,y) + c \qquad (2)$$

k – the Darcy hydraulic conductivity
h – piezometric head of aquifer
c – an undetermined constant.

The complex potential of the flow F(z) has the property

$$\text{Re}\{F(z)\} = \phi(x,y), \qquad (3)$$

$$\text{Im}\{F(z)\} = \psi(x,y), \quad z \in D^- \qquad (4)$$

where Ψ(x,y) is the stream function

The balance equation of steady groundwater flow:

$$\text{div}\,\vec{v} = \nabla \cdot \vec{v} = 0 \qquad (5)$$

Both functions Φ(x,y) and Ψ(x,y) satisfied the Laplace equation in the flow domain D^-:

$$\Delta\Phi = 0 \text{ and } \Delta\Psi = 0 \qquad (6)$$

where Δ is the Laplace operator

Consequently, for mathematical representations of the groundwater flow can be used one of analytical functions of a complex variable F(z) or W(z). The complex plane (z) is called also as physical plane of the considered groundwater flow.

Further on the complex velocity function W(z) as basic solution for the considered groundwater flow will be determined.

The mathematical relationship between the functions F(z) and W(z) is given as:

$$W(z) = \frac{dF(z)}{dz} = v_x - iv_y, \; z \in D^- \qquad (7)$$

v_x şi v_y are the components of flow velocity.

Consequently, for mathematical representation of the groundwater flow it is enough to know one of the functions ∅ (x, y), F (z) or W (z).

Further on in the present paper the velocity potential W(z) will be determined as first and after that the other both functions (i.e. the complex potential F(z) and its real part the potential function ∅ (x, y). Knowing ∅ (x, y) the velocity distribution and the cavity discharge can be determine using the relations (1) and (2) and as well the given piezometric head in the cavity and at the influence range of cavity.

To determine of the complex velocity W(z) boundary conditions are necessary which assure the uniqueness of the solution function:

- W(z) should be a holomorphic

function of a complex variable in D^-. This condition can be expressed as:

$$\frac{\partial W(z)}{\partial \bar{z}} = 0, \quad z \in D^-, \quad z=x+iy, \quad \bar{z}=x-iy \qquad (8)$$

- The permeable part C_0 of the cavity contour C is an equipotential line i.e. the tangential component V_τ of the flow velocity on C_0 is 0:

$$v_\tau = \frac{\partial\phi}{\partial\tau} = \text{Re}\{W(z)dz\} = 0, \; z \in C_0 \qquad (9)$$

The impermeable part C_Σ of the cavity contour C is a streamline i.e. the normal component v_n of the velocity on C_Σ is 0:

$$v_n = \frac{\partial\phi}{\partial n} = -\text{Im}\{W(z)dz\} = 0, \; z \in C_\Sigma \qquad (10)$$

The asymptotic condition i.e. the groundwater flow velocity at large distances from the cavity is equal to undisturbed groundwater flow velocity and is expressed as (Fig. 1):

$$\lim_{z\to\infty} W(z) = W_\infty = V_0\, e^{-i\alpha} \qquad (11)$$

The recharge/discharge flow rate Q (i.e. injection or extraction flow rate) of the cavity is constant expressed as:

$$\int_{C_0} W(z)dz = \pm iQ \qquad (12)$$

To determine the complex velocity W(z) which satisfied the above-mentioned conditions a new complex plan (ζ) will be considered. In this plane the cavity C become a canonical form i.e. a circle K having a radius ρ_0 (Fig.2a, b). The flow domain D^- from the physical complex plane

(z) will be conform mapped on the exterior domain Δ^- of the circular cavity K in the complex plan (ς) (Fig.2 a,b):

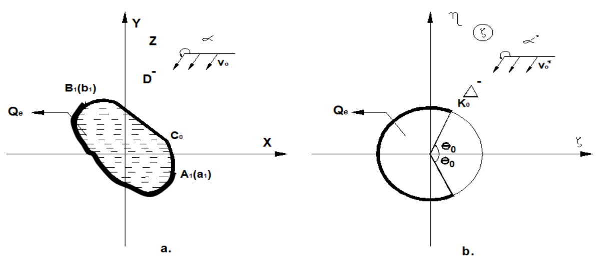

Figure 2. Scheme of conformal mapping of the flow domain in the complex plane (z) on the external domain of a circle in the complex plan (ς)

This mapping can be made using holomorphic function of a complex variable and is called conformal mapping [1], [4], [5]:

$$\zeta = f(z) \qquad (13)$$

This function is invertible having an inverse:

$$z = f^{-1}(\zeta) \qquad (13')$$

So we have the following correspondence between the complex plans (z) and (ς):

$$\begin{aligned} & z \leftrightarrow \zeta \\ & D^{-} \leftrightarrow \Delta^{-} \\ & C_0 \leftrightarrow K_0 \\ & C_\Sigma \leftrightarrow K_\Sigma \end{aligned} \qquad (14)$$

The relationship between the complex flow velocity in both complex plans, original physical complex plane (z) and her image plane (ζ):

$$W(\zeta) = W(z)\frac{dz}{d\zeta} \quad \text{and} \quad W(z) = W(\zeta)\frac{d\zeta}{dz} \qquad (15)$$

On the contours C and K is valid:

$$\lim_{z \in D^{-} \to z_C \in C} W(z) = W(z_C) \quad \text{and}$$

$$\lim_{\zeta \in \Delta^{-} \to \varsigma_K \in K} W(\zeta) = W(\varsigma_K) \qquad (16)$$

With these correspondences between both complex plans we can rewrite the boundary conditions (8)-(12) for the complex velocity W (ζ) as follow:

$$\frac{\partial W(\zeta)}{\partial \bar{\zeta}} = 0, \quad \zeta \in \Delta^{-} \qquad (8')$$

$$v_{\varsigma_0} == \mathrm{Re}\left\{W(\varsigma_0)\, d\xi\right\} = 0,$$

$$\varsigma_0 \in K_0 \qquad (9')$$

$$v_n = -\mathrm{Im}\left\{W(\varsigma_\Sigma)\, d\xi\right\} = 0,$$

$$\varsigma_\Sigma \in K_\Sigma \qquad (10')$$

and

$$\int_{K_0} W(\zeta)\, d\zeta = -iQ$$

(12')

The asymptotic condition is expressed as

$$W_\infty^* = \lim_{\zeta \to \infty} W(\zeta) = \lim_{z \to \infty} W(z)\left(\frac{dz}{d\zeta}\right) =$$

$$= b_0 W_\infty = b_0 V_0\, e^{-i\alpha} \qquad (11')$$

where

$$b_0 = \left(\frac{dz}{d\zeta}\right)_{z \to \infty}$$

In the complex plane (ς) in which the cavity C became a circle K as canonic cavity one can be determine the complex velocity W(ζ) using Cauchy integral theory for holomorphic functions in the same way as in [1], [4] applied for a cavity with entire permeable contour.

By means of mathematical developments, which are not reproduced here, the following mathematical expression is finally obtained for the function W(ς):

$$W(\zeta) = \frac{1}{A(\varsigma, \rho_0, \theta_0)} \begin{bmatrix} W_\infty^*(\zeta - \rho_0 \cos\theta_0) + \\ \dfrac{\rho_0^2}{\zeta^2}\overline{W_\infty^*}(\rho_0 - \zeta\cos\theta_0) - \\ \dfrac{Q_e}{2\pi}\left(1 + \dfrac{\rho_0}{\zeta}\right) \end{bmatrix} \qquad (16)$$

where

$$A(\varsigma, \rho_0, \theta_0) = \sqrt{\zeta^2 - 2\zeta\rho_0\cos\theta_0 + \rho_0^2}$$

$$W_\infty^* = W_\infty \cdot \left(\frac{dz}{d\zeta}\right)_\infty \quad and \quad \rho_0 = \left|f(z \in C)\right| \qquad (17)$$

By means of the conformal mapping (13), (13') and (15) from W(ς) can be obtain the complex velocity in the physical plane W(z) by replacing ς with f(z) resulting:

$$W(z) = W(f(z))\frac{d\varsigma}{dz} = \frac{1}{A(z, \rho_0, \theta_0)} \cdot$$

$$\cdot \left\{ \begin{aligned} & W_\infty^*\left[f(z) - \rho_0 \cdot \cos\theta_0\right] + \\ & \frac{\rho_0^2}{f(z)^2} \cdot \overline{W_\infty^*} \cdot \left[\rho_0 - f(z) \cdot \cos\theta_0\right] \\ & -\frac{Q_e}{2\pi} \cdot \left(1 + \frac{\rho_0}{f(z)}\right) \end{aligned} \right\} \frac{df(z)}{dz} \qquad (18)$$

where

$$A(z, \rho_0, \theta_0) = \sqrt{f(z)^2 - 2 \cdot \rho_0 \cdot f(z) \cdot \cos\theta_0 + \rho_0^2}$$

The complex velocity W(z) is the searched basic mathematical representations of the considered groundwater flow in shallow aquifer containing a cavity bounded with a counter C which consist of two parts, a permeable part C_0 and an impermeable part C_Σ (Fig. 1).

From W(z) one obtain the complex potential F(z) using the relationships (7) between these functions:

$$F(z)=\left[W_\infty^* - \overline{W_\infty^*}\cdot\frac{\rho_0}{f(z)}\right]\cdot A(z,\rho_0,\theta_0)$$
$$-\frac{Q_e}{2\pi}\cdot ln\frac{f(z)\cdot\left[A(z,\rho_0,\theta_0)+f(z)-\rho_0\cdot cos\,\theta_0\right]}{\rho_0\left[A(z,\rho_0,\theta_0)\;-f(z)\cdot cos\,\theta_0+\rho_0\right]} \quad (19)$$

where

$$A(z,\rho_0,\theta_0)=\sqrt{f(z)^2-2\cdot\rho_0\cdot f(z)\cdot cos\,\theta_0+\rho_0{}^2}$$

$$W_\infty^*=b_\infty V_0\cdot e^{-i\cdot\alpha}\quad where\; b_0=\left(\frac{dz}{d\zeta}\right)_\infty$$

$\overline{W_\infty^*}$ *is complex conjgate of* W_∞^*

From the complex potential one can be determine its real part i.e. the flow potential function Φ(x,y) which allows the determination of hydraulic flow parameters like velocity distribution, discharge/recharge rate of the cavity (i.e. Q_e/Q_i) and so on.

In the next paragraph an example will be discussed.

It is to mention also that for $\theta_0=\pi$ the mathematical representations (18) and (19) coincide with results discussed in [1] in the case of a cavity with fully permeable contour ($C=C_0$) which is thereby a simplified case of that considered in the present paper:

$$F(z)=\left[W_\infty^* - \overline{W_\infty^*}\cdot\frac{\rho_0}{f(z)}\right]\cdot$$
$$-\frac{Q_e}{2\pi}\cdot ln\frac{f(z)}{\rho_0} \quad (19')$$

where

$$W_\infty^*=V_0\cdot e^{-i\cdot\alpha}\cdot\left(\frac{dz}{d\zeta}\right)_\infty\; and$$

$\overline{W_\infty^*}$ *is complex conjgate of* W_∞^*

Knowing the complex potential of the groundwater flow, one can proceed to determine the calculation formula for the most important hydraulic parameters like discharge/recharge rate of cavity Q_e/Q_i).

3. EXAMPLE

A drainage trench of length 2L, depicted in Fig.3, will be as example considered. The impermeable portion of the trench is situated on its downstream side and has the same length of 21_e =2L. The drainage trench is placed in a parallel groundwater flow (i.e. α=π).

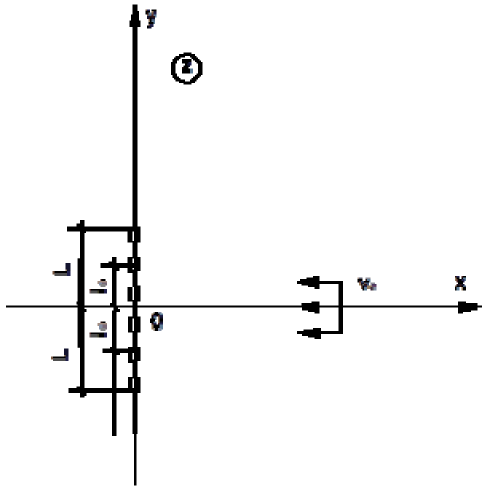

Figure 3. Scheme of drainage trench disposed in a parallel groundwater flow

The conformal mapping function (13) has the form

$$\zeta=f(z)=z+\sqrt{z^2+L^2} \quad (21)$$

Consequently,

$$z=\frac{1}{2}(\varsigma-\frac{L^2}{\varsigma})\rightarrow\left(\frac{dz}{d\varsigma}\right)_\infty=\frac{1}{2}$$
$$W_\infty^*=V_0\cdot e^{-i\cdot\alpha}\cdot\left(\frac{dz}{d\zeta}\right)_\infty=-\frac{1}{2}V_0$$
$$cos\,\theta_0=\pm\sqrt{1-(\frac{l_e}{\rho_0})^2} \quad (22)$$
$$\rho=\left|f(z)\right|\quad\rightarrow z=0,\;\;\rho=\rho_0=L$$

Along the ox axis is valid

$$\rho=\left|f(z=x)\right|=x+\sqrt{x^2+L^2}$$

For calculation of the discharge rate it is enough to know the potential function Φ (i.e. the real part of the complex potential F(z)) along of the ox axis (y=0). Considering these particularities, the potential function along the x axis is expressed as

$$\Phi(x, y=0) = -V_0 \cdot \left(1 - \frac{\rho_0}{\rho}\right) \cdot A(\rho, \rho_0, \theta_0) -$$

$$\frac{Q_e}{2\pi} \cdot \ln \frac{\rho}{\rho_0} \cdot \frac{A(\rho, \rho_0, \theta_0) + \rho - \rho_0 \cdot \cos\theta_0}{A(\rho, \rho_0) - \rho \cdot \cos\theta_0 + \rho_0} \quad (23)$$

where

$$A(\rho, \rho_0, \theta_0) = \sqrt{\rho^2 - 2 \cdot \rho_0 \cdot \rho \cdot \cos\theta_0 + \rho_0^{\ 2}}$$

Knowing the potential function $\Phi(x,y)$ one can proceed to determine the calculation formula for the most important hydraulic size of the catches which is the discharge (i.e. extraction) or recharge (i.e. infiltration) rate of cavity (Q_e/Q_i). For this purpose, the hydraulic boundary conditions will be used (see relationship 2):

$h = H_0$ the piezometric head in the drain (x=0) implying the relationship

$$\rho = \rho_0 = \left| f(z=0) \right| = L \quad (24)$$

$h=H_R$ the piezometric head at the influence range x=R of the drain implying the relationship

$$\rho = \rho_R = \left| f(z=R) \right| = R + \sqrt{R^2 + L^2} \quad (24^*)$$

The parallel flow velocity V_0 can be also expressed with the help of the groundwater slope I_0 and the hydraulic conductivity k:

$$V_0 = kI_0 \quad (25)$$

Th piezometric height difference in the drain will be denoted as

$$H_R - H_0 = \Delta H \quad (26)$$

Using the notations above from the potential function (23) one can express the drainage trench flow rate Q_e as follow:

$$\frac{Q_e}{k \cdot T \cdot \Delta H} = \frac{2\pi\left(1 - \frac{I_0 R}{\Delta H} \cdot U_e\right)}{\ln G} \quad (27)$$

where

$$U_e = \frac{1}{2R} \cdot \frac{\rho_R - \rho_0}{\rho_R} \cdot A(\rho_R, \rho_0, \theta_0) \quad (28)$$

$$G = \frac{\rho_R}{\rho_0} \cdot \frac{A(\rho_R, \rho_0, \theta_0) + \rho_R}{A(\rho_R, \rho_0, \theta_0) + \rho_0}$$

A special case is when the impermeable part extends on entire length of the drainage trench i.e. l_e=L. Consequently in (27), (28) $\theta_0=\pi/2$ and are valid the following relationships.

$$A(\rho_R, \rho_0, \theta_0) = A(\rho_R, \rho_0, \theta_0 = \pi/2) =$$

$$A(\rho_R, \rho_0) = \sqrt{\rho_R^{\ 2} + \rho_0^2}$$

$$U_e = \frac{1}{2R} \cdot \frac{\rho_R - \rho_0}{\rho_R} \cdot A(\rho_R, \rho_0) \quad (28')$$

$$G = \frac{\rho_R}{\rho_0} \cdot \frac{A(\rho_R, \rho_0) + \rho_R}{A(\rho_R, \rho_0) + \rho_0}$$

From these expressions on can obtained the case of the drain trench with entire permeable contour (i.e. $2l_e=0$) discused discussed in [1]:

$$\frac{Q_e}{k \cdot T \cdot \Delta H} = \frac{2\pi\left(1 - \frac{I_0 R}{\Delta H} \cdot U\right)}{\ln G} \quad (27'')$$

where

$$U = 1 \ \ and \ G = \frac{\rho_R}{\rho_0} = \frac{R}{L} + \sqrt{\frac{R^2}{L^2} + 1} \quad (28'')$$

Comparing (27) and (27'') one cane analyse the influence of the impermeable length ($2l_e$) of the drainage trench. For a comparative calculation, the ratio of the discharge rate of both drainage trenches Q_e/Q is considered (Q_e for the drainage trench with impermeable part and Q for the entire permeable drain trench) can be expressed from (27) and (27') as follow:

$$\frac{Q_e}{Q} = \frac{1 - \frac{I_0 R}{\Delta H} \cdot U_e}{1 - \frac{I_0 R_0}{\Delta H} \cdot U} \cdot \frac{ln \frac{\rho_R}{\rho_0}}{ln G} \quad (29)$$

As representative example of this ratio is depicted in Fig. 4 as function of drainage trench relative influence range (λ=R/L)

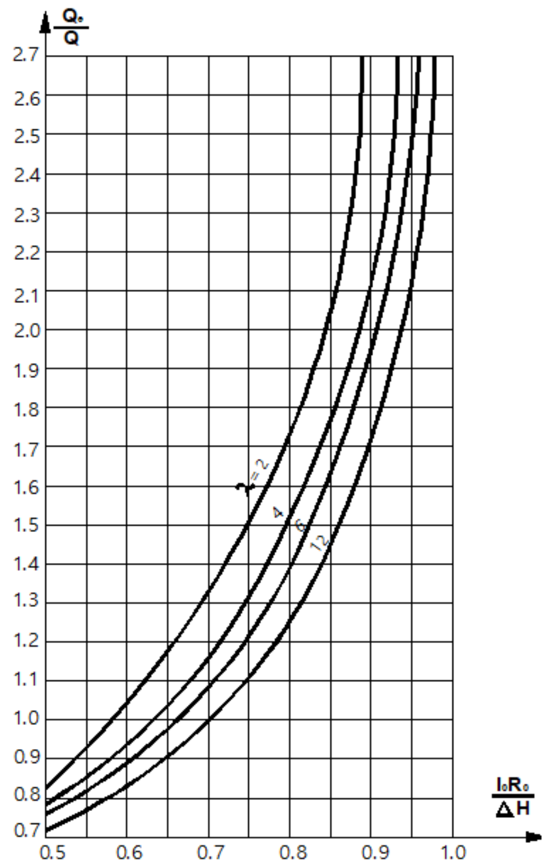

Figure 4. Discharge rate ratio Q_e/Q of both drainage trenches: with impermeable downstream side (l_e=L) and entire permeable drain (l_e=0)

On can see, that the impermeabilized downstream side (l_e=L) can strongly influence of the discharge rate especially for increased slope of the groundwater flow.

This result is important for the practical case of an interception drainage trench provided for the defence of a construction objective located downstream of the drain.

4. CONCLUSIONS

In the present paper an analytical solution, based calculus formula for plane groundwater flow containing an extraction/infiltration cavity of arbitrary shape, bounded with contour which contains an impermeable part is presented.

There are deduced mathematical representation of the groundwater flow using the complex functions theory. These representations refer to the complex velocity, the complex potential and the potential. Based on these representations a general formula for calculus of the discharge rate is deduced for groundwater flow system, consisting of a cavity of arbitrary shape, bounded with a counter which contain an impermeable part. The cavity is situated in a pre-existing parallel groundwater. These representations generalized those obtained in a former paper related to a cavity bounded with an entire permeable contour.

The solutions obtained for an extraction cavity of arbitrary form can be applied for several practical shapes which currently are used in technical applications. In the paper a drainage trench of length 2L, depicted in Fig.3, as representative example was considered. The impermeable portion of the trench is situated on its downstream side and has the same length of 21_e. The general representations as well as the discharge calculus formula are particularized for this cavity shape.

As numerical example a comparative analyse is performed comparing the discharge rate of a drainage trench with an impermeable side and of an entire permeable drainage trench as function of drainage trench influence radius.

In paper obtained results can used in engineering planning and management of a large number of groundwater flow problems like groundwater balance in ecologic lakes, ponds, groundwater recharge pits, drainage pits, foundation pits, wells or wells with laterals etc. which can be modelled as extraction cavities or recharge cavities. For each practical case first must determine the complex mapping function which mapped

the given cavity on the canonic cavity (i.e. a circle). Knowing this function one cane expressed all analytical representations and calculus formulas for the considered groundwater flow problem.

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