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ANALYTICAL AND NUMERICAL ANALYSIS OF VELOCITY AND TEMPERATURE FIELDS FOR LAMINAR FORCED HEAT CONVECTION IN CONCENTRIC ANNULAR TUBE

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Abstract: This paper presents a study of the dual reciprocity boundary element method (DRBEM) for the laminar heat convection problem between two coaxial cylinders with constant heat flux boundary condition. DRBEM is one of the most successful techniques used to transform the domain integrals arising from the nonhomogeneous term of the Poisson equation into equivalent boundary only integrals.
Keywords: Concentric annular tube, Laminar heat convection, Heat flux boundary condition, Numerical analysis, Dual reciprocity boundary element method.

1. INTRODUCTION

Among the various numerical methods, the boundary element method (BEM) becomes one of the favorite analysis tools ever since its introduction to the solution of heat transfer problems. Its advantage over the finite difference or the finite element methods comes from the fact that instead of full domain discretization, only the boundary is discretized into elements and internal point position can be freely defined. Therefore the quantity of data necessary to solve the problems can be greatly reduced [2].

Until recent years the main area of the BEM application has been limited to the conduction heat transfer problems among different heat transfer modes and therefore, with various research efforts, BEM for the solution of heat conduction direct or inverse problem is now well established [3]. However BEM study for the application of heat convection problems can be considered as insufficient and in still developing stage. Since the convection effects are of considerable importance in many heat transfer phenomena, they need much more research focus. The main difficulties of the BEM application to such problems are due to the facts that the fundamental solutions are only available for the few governing equation types and, except Laplace equation, additional domain discretization is required to account source type domain integral terms.

The dual reciprocity boundary element method (DRBEM) which was introduced by Nardini and Brebbia [5] is thus far the most successful technique for dealing with above mentioned lack of fundamental solution types and domain integral problems.

2. FORMULATION OF THE PROBLEM

Consider an incompressible Newtonian fluid flow in a concentric annular tube as shown in Figure 1. In the system to be analyzed, z coordinate represents the axial direction and x–y coordinates are attached to the cross–sectional surface. The inner and outer cylinder radii are taken as R_i and R_o.

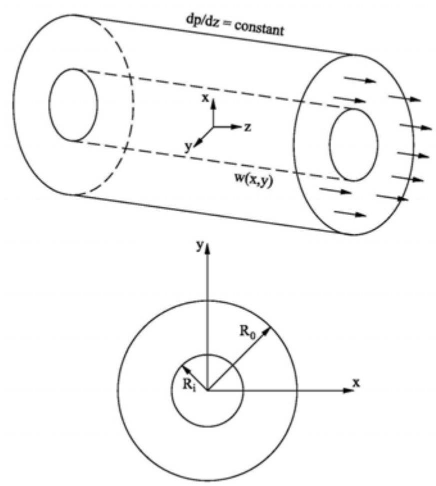

Fig. 1 – Geometry of the concentric annular tube

For the fully developed steady laminar flow with constant transport properties and negligible body forces, Navier–Stokes equation becomes simple pressure driven Poiseuille flow equation. Since the flow is fully developed, axial flow velocity is a function of only x–y coordinates, and axial pressure gradient is constant. In the energy equation, the viscous dissipation and axial heat conduction effects are neglected. Therefore the governing equation can be expressed in the form of a Poisson equation as follows:

– momentum equation:

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$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z} \tag{1}$$

– energy equation:

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{w}{a}\frac{\mathrm{d}T}{\mathrm{d}z} \tag{2}$$

in which: w is the axial flow velocity; μ – coefficient of viscosity; p – pressure; T – temperature; $a=\lambda/\rho c$ – thermal diffusivity.

For the thermally fully developed flow with constant heat flux boundary condition, equation (2) can be rewritten by using the mixed mean temperature T_m [4] as:

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{w}{a}\frac{\mathrm{d}T_m}{\mathrm{d}z} \tag{3}$$

where $\partial T/\partial z = \mathrm{d}T_m/\mathrm{d}z =$const. from the given conditions. The boundary conditions associated with the equations (1) and (3) are:

$$w = 0 \text{ at } R = R_i, \quad w = 0 \text{ at } R = R_\mathrm{o} \tag{4}$$

$$T = T_i \text{ at } R = R_i, \quad T = T_\mathrm{o} \text{ at } R = R_\mathrm{o} \tag{5}$$

where subscripts "i" and "o" represent for the inner and outer surfaces.

For the solution of temperatures, velocity from equation (1) is obtained first and then equation (3) can be solved in sequence since the assumption of constant viscosity uncoupled the momentum and energy equations.

3. DUAL RCIPROCITY BOUNDARY ELEMENT EQUATION

For the BEM solution, equations (1) and (3) subject to equations (4) and (5) can be generalized as the following type of Poisson equation [6]:

$$\nabla^2 u(x,y) = b(x,y), \qquad (x,y) \in \Omega \tag{6}$$

with the boundary conditions:

$$u(x,y) = \bar{u}, \qquad (x,y) \in \Gamma_1 \tag{7}$$

$$q(x,y) = \frac{\partial u(x,y)}{\partial n} = \bar{q}, \quad (x,y) \in \Gamma_2 \tag{8}$$

and to represent convective heat transfer problems:

$$\begin{aligned} u(x,y) &= w, \quad b(x,y) = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z} = \text{const.} \\ u(x,y) &= T, \quad b(x,y) = \frac{w}{a}\frac{\mathrm{d}T_m}{\mathrm{d}z} \end{aligned} \tag{9}$$

where: $\Gamma_1+\Gamma_2=\Gamma$ is the total boundary of solution domain Ω; n – normal to the boundary; $\bar{u}$ and $\bar{q}$ – specified values at each boundary.

Applying the usual boundary element technique to equation (6), an integral equation can be deduced as follows:

$$c_i u_i + \int_\Gamma u q^* \mathrm{d}\Gamma - \int_\Gamma q u^* \mathrm{d}\Gamma = \int_\Omega b u^* \mathrm{d}\Omega \tag{10}$$

where the constant c_i depends on the geometry at point i as:

$$c_i = \begin{cases} 1 & \text{for } (x_i, y_i) \in \Omega \\ \dfrac{\theta}{2\pi} & \text{for } (x_i, y_i) \in \Gamma \end{cases} \tag{11}$$

where θ is the angle between the tangent to Γ on either side of point i.

The key method of DRM is to take the domain integral of equation (10) to the boundary and remove the needs of complicated domain discretization. To accomplish this idea, the source term $b(x, y)$ is expanded as its values at each node j and a set of interpolating functions f_j are used as [6]:

$$b(x,y) \cong \sum_{j=1}^{N+L} \alpha_j f_j \tag{12}$$

where: α_j is a set of initially unknown coefficients; $N+L$ – the number of boundary nodes plus internal points.

If the function $\bar{u}_j$ can be found such that:

$$\nabla^2 \bar{u}_j = f_j \tag{13}$$

then the domain integral can be transferred to the boundary.

Substituting equation (13) into equation (12), and applying integration by parts twice for the domain integral term of equation (10) leads to:

$$\begin{aligned} & c_i u_i + \int_\Gamma u q^* \mathrm{d}\Gamma - \int_\Gamma q u^* \mathrm{d}\Gamma = \\ & \sum_{j=1}^{N+L} \alpha_j \times \left(c_i \hat{u}_{ij} + \int_\Gamma \hat{u}_j q^* \mathrm{d}\Gamma - \int_\Gamma \hat{q}_j u^* \mathrm{d}\Gamma \right) \end{aligned} \tag{14}$$

For the two dimensional domain of interest in this study, u^*, q^* and $\hat{u}$, $\hat{q}$ can be derived as:

$$\begin{aligned} u^* &= \frac{1}{2\pi}\ln\left(\frac{1}{r}\right) \\ q^* &= \frac{-1}{2\pi r}\nabla r \cdot \vec{n} \end{aligned} \tag{15}$$

$$\begin{aligned} \hat{u} &= \frac{r^2}{4} + \frac{r^3}{9} \\ \hat{q} &= \left(\frac{r}{2} + \frac{r^2}{3}\right)\nabla r \cdot \vec{n} \end{aligned} \tag{16}$$

where r stands for the distance from a source point i or a DRM collocation point j to a field point (x, y). As for the equation (13), a radial basis function $f = 1+r$ is chosen as an interpolating function which was shown to be generally sufficient.

In the numerical solution of the integral equation (14), u, q, $\hat{u}$ and $\hat{q}$ in the integrals are modelled using the linear interpolation functions as:

$$\int_{\Gamma_k} uq^* \mathrm{d}\Gamma = u_k h_{ik}^1 + u_{k+1} h_{ik}^2 \qquad (17)$$

$$\int_{\Gamma_k} qu^* \mathrm{d}\Gamma = q_k g_{ik}^1 + q_{k+1} g_{ik}^2 \qquad (18)$$

$$\int_{\Gamma_k} \hat{u}_j q^* \mathrm{d}\Gamma = \hat{u}_{kj} h_{ik}^1 + \hat{u}_{(k+1)j} h_{ik}^2 \qquad (19)$$

$$\int_{\Gamma} \hat{q}_j u^* \mathrm{d}\Gamma = \hat{q}_{kj} g_{ik}^1 + \hat{q}_{(k+1)j} g_{ik}^2 \qquad (20)$$

where:

$$h_{ik}^1 = \int_{\Gamma_k} \Phi_1 q^* \mathrm{d}\Gamma, \quad h_{ik}^2 = \int_{\Gamma_k} \Phi_2 q^* \mathrm{d}\Gamma \qquad (21)$$

$$g_{ik}^1 = \int_{\Gamma_k} \Phi_1 u^* \mathrm{d}\Gamma, \quad g_{ik}^2 = \int_{\Gamma_k} \Phi_2 u^* \mathrm{d}\Gamma \qquad (22)$$

Here the first subscript of Eq. (21) and (22) refers to the specific position of the point where the flow velocity or temperature is evaluated; the second subscript refers to the boundary element over which the contour integral is carried out. The superscript 1 and 2 designate the linear interpolation function Φ_1 and Φ_2 respectively, with which the u^* and q^* functions are weighted in the integrals in equation (17) through (20).

For the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ discretized into N elements, integral terms in equation (14) can be rewritten as:

$$\int_{\Gamma} uq^* \mathrm{d}\Gamma = \sum_{k=1}^{N} \int_{\Gamma_k} uq^* \mathrm{d}\Gamma = \sum_{k=1}^{N} \left[h_{i(k-1)}^2 + h_{ik}^1\right] u_k = \sum_{k=1}^{N} H_{ik} u_k \text{ or } = \sum_{j=1}^{N_n} H_{ik} u_{kj} \text{ for } \hat{u}_j \qquad (23)$$

$$\int_{\Gamma} qu^* \mathrm{d}\Gamma = \sum_{k=1}^{N} \int_{\Gamma_k} qu^* \mathrm{d}\Gamma = \sum_{k=1}^{N} \left[g_{i(k-1)}^2 + g_{ik}^1\right] q_k = \sum_{k=1}^{N} G_{ik} q_k \text{ or } = \sum_{j=1}^{N_n} G_{ik} \hat{q}_{kj} \text{ for } \hat{q}_j \qquad (24)$$

where $h_{i0}^2 = h_{iN}^2$ and $g_{i0}^2 = g_{iN}^2$. Introducing equation (23) and (24) into equation (14) and manipulating results yields a dual reciprocity boundary element equation as:

$$c_i u_i + \sum_{k=1}^{N} H_{ik} u_k - \sum_{k=1}^{N} G_{ik} q_k = \sum_{j=1}^{N+L} \alpha_j \times \left(c_i \hat{u}_{ij} + \sum_{k=1}^{N} H_{ik} \hat{u}_{kj} - \sum_{k=1}^{N} G_{ik} \hat{q}_{kj} \right)$$

(25)

4. NUMERICAL SOLUTION

For the computer implementation of numerical solution, equation (25) can now be written in a matrix form as:

$$\mathbf{HU} - \mathbf{GQ} = (\mathbf{H\hat{U}} - \mathbf{G\hat{Q}})\boldsymbol{\alpha} \qquad (26)$$

where **H** and **G** are matrices of their elements being H_{ik} and G_{ik}, with c_i being incoroperated into the principal diagonal element, respectively. **U**, **Q** and their terms with hat of equation (26) correspond to vectors of u_k, q_k and matrices with j the column vectors of hat u_{kj}, q_{kj}. It is noted that vector $\boldsymbol{\alpha}$ of unknown coefficients j can be evaluated from equation (12) with chosen interpolating function f_j and the function $b(x, y)$ of governing equation. Therefore introducing the boundary conditions into the nodes of u_k and q_k vectors and rearranging by taking known quantities to the right hand side and unknowns to the left hand side leads to a set of simultaneous linear equations of the form:

$$\mathbf{AX} = \mathbf{B} \qquad (27)$$

Consider the geometry illustrated in Figure 2. For the sake of simplification, the surface temperatures of two cylinders are assumed to be equal. Thus, the solution satisfies the following boundary conditions:

$$w(x,y)\big|_{R=R_i} = w(x,y)\big|_{R=R_\mathrm{o}} = 0$$

$$T^*(x,y)\big|_{R=R_i} = T^*(x,y)\big|_{R=R_\mathrm{o}} = 0$$

$$T^* = T_w - T, \quad T_w = T_i = T_\mathrm{o}$$

(28)

For the numerical test case, following numerical values in equations (1) and (3) are taken from the paper [8] where the spectral collocation method is used for the exccentric annuli heat convection analysis: R_o=0.055 m, R_i=0.030 m, $(1/\mu)\mathrm{d}p/\mathrm{d}z$ = -836 $\mathrm{m}^{-1}\cdot\mathrm{s}^{-1}$, a=1.3418×10^{-9} m^2/s, $\mathrm{d}T_m/\mathrm{d}z$=0.47 °C/m. The numerical model develobed above, based on DRBEM, was implemented by the author in programs DISCRMM, DRMMVTP and DRMMPTE elaborated in FORTRAN programming language, for IBM–PC compatible computers.

5. RESULTS AND DISSCUTION

In order to confirm the accuracy of the dual reciprocity boundary element method for the present heat convection problem, each boundary of outer and inner surface is equally discretized as 36, 48, 60, 72 and 84 elements respectively. The nodes on every boundary and the unternal points of the analysis domain are located as shown in Figure 2. Therefore total number of internal points used in the analysis are: 90, 120, 150, 180 and 210 for each 36, 48, 60, 72 and 84 boundary element cases respectively.

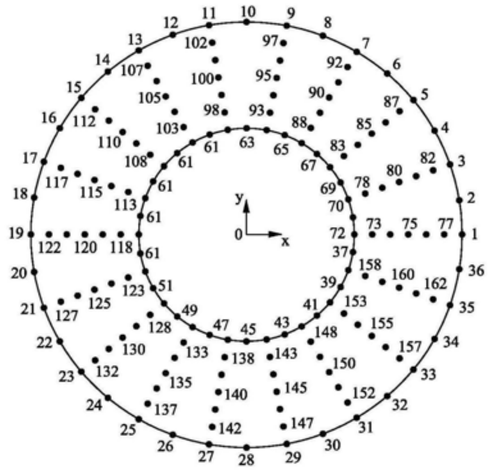

Fig. 2 – Boundary element nodes and internal points for the system to be analyze

Table 1. DRBEM results with exact solutions for the boundary and internal locations in flow velocity analysis

Solution Variable	Radial location R [m]	DRBEM solution (number of boundary elements case)					Exact solution
		36	48	60	72	84	
$\partial w/\partial n$	0.055	−9.570611	−9.61436	−9.632446	−9.64673	−9.64945	−9.66790
$\partial w/\partial n$	0.030	−11.9614	−11.9317	−11.91098	−11.9023	−11.90070	−11.8838
W	0.0342	0.040336	0.039937	0.039755	0.039656	0.039614	0.039413
W	0.0383	0.061518	0.061112	0.060926	0.060825	0.060760	0.060591
W	0.0425	0.066727	0.066320	0.066133	0.066030	0.065973	0.065803
W	0.0466	0.057611	0.057196	0.057006	0.056908	0.056853	0.056682
W	0.0508	0.035084	0.034883	0.034733	0.034638	0.034606	0.034439

Table 2. DRBEM results with exact solutions for the boundary and internal locations in temperature analysis ($T^*=T_w-T$)

Solution variable	Radial location R [m]	DRBEM solution (number of boundary elements case)					Exact solution
		36	48	60	72	84	
$\partial T^*/\partial n$	0.055	172056.38	170824.22	171362.75	171600.75	171628.56	170484.2
$\partial T^*/\partial n$	0.030	233420.45	229217.02	230340.65	230698.70	230771.22	229506.9
T^*	0.0342	804.26	828.59	847.40	853.78	855.04	858.72
T^*	0.0383	1268.86	1298.52	1311.65	1315.86	1317.26	1320.13
T^*	0.0425	1383.48	1413.28	1426.46	1429.68	1431.26	1433.69
T^*	0.0466	1180.68	1209.26	1224.48	1230.46	1231.92	1234.96

To obtain the axial flow velocity distribution $w(x,y)$, equation (1) is solved first. Their results for the boundary and internal nodes are shown in Table 1. Here the normal derivative of velocity w at the boundary is listed as well, and all the numerical solutions are compared with the exact solutions [4] for their accuracy. Considering relation (17.8) from the bibliographic reference[17] proposes for the analytical calculus of velocity w the equation:

$$w = -\frac{dp}{4\eta dx}\left[R_i^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln(\frac{R_o}{R_i})}\ln(\frac{r}{R_i})\right] \quad (29)$$

Figures 3 and 4 show the convergence plot of DRBEM velocity and its normal derivative solutions as the number of boundary elements and internal points increase. DRBEM solutions are in close agreement with the exact solutions and relative errors are within 2.3% for the above 36 element cases.

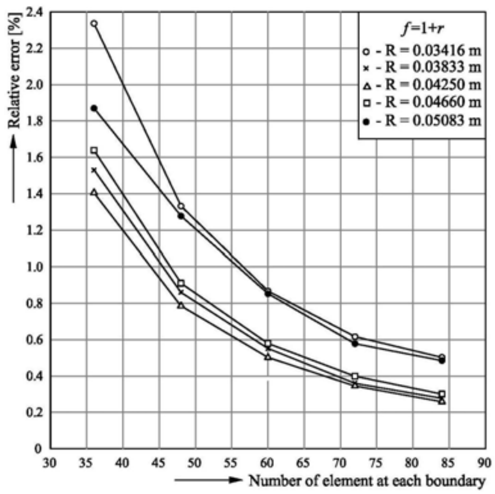

Fig. 3 Accuracy test for the velocity solution at the selected internal points.

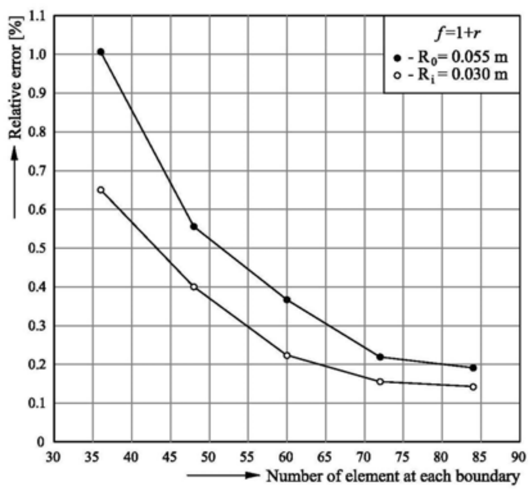

Fig. 4 Accuracy test for the normal derivative of velocity solution at the inner and outer boundaries.

As noted in Figure 3, velocity solutions at location of R=0.0508 m and R=0.0342 m are less accurate than the others and, in between, R=0.0342 point gives more inaccurate solution than R=0.0508. And for the normal derivatives of velocity at boundary R=0.055 m is less accurate than R=0.030 m as shown in Figure 4. These results are due to the facts that the outer boundary element size is larger than the inner boundary element size and distribution of internal points is getting sparse to the outward direction, whereas rapid change of velocity occurs at inner and outer boundary sides as illustrated in Figures 2 and 5. Therefore solution's error magnitude regarding to the radial location is closely related to both the physical and the mathematical aspects and nevertheless overall solution accuracy is shown to be fairly acceptable. Thus, 36 element solution case shows maximum 2.34% error at radial position R=0.0342 and later results in accurate temperature solution.

Then these DRM velocity solutions are, in turn, used in the energy equation (3) to solve for the temperature distribution. Table 2 shows the results, and it is found that DRM solutions are in excellent agreement with exact solutions and relative errors are within 5% for the above 36 element cases (Fig. 6, 7 and 8). Although the converging trend in Figure 7 is not monotonic and radial location effect about error magnitude is not exactly following the previously discussed velocity solution case, solution trends can be considered as indistinguishable within 1% relative error.

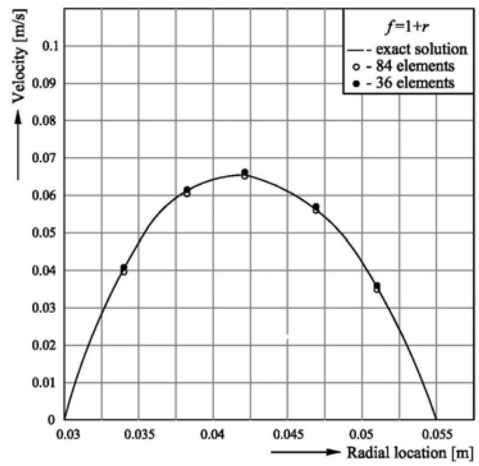

Fig. 5 Velocity profile of exact solution compared with DRBEM results.

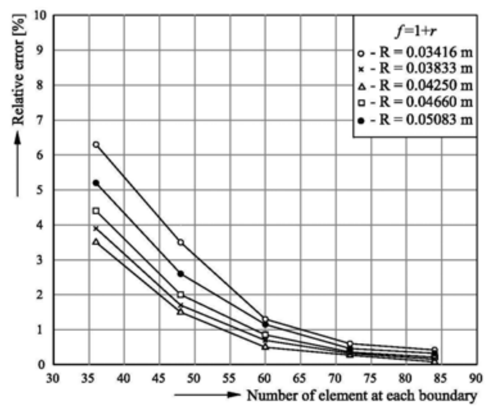

Fig. 6 – Accuracy test for the temperature solution at the selected internal points.

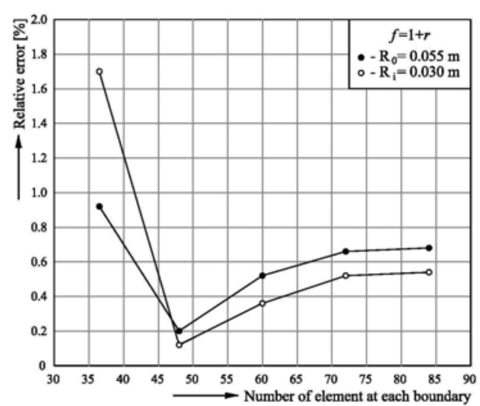

Fig. 7 Accuracy test for the normal derivative of temperature solution at the inner and outer boundaries.

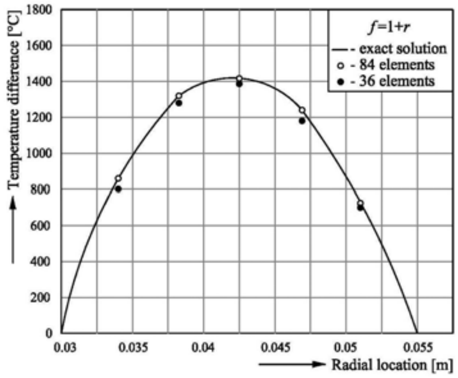

Fig. 8 – Temperature profile T^* of exact solution compared with DRMEM results.

As a final note, all the element cases turns out to be adequate for the solution of this problem. Errors of the velocity and the temperature solution are acceptable.

6. CONCLUSIOS

A dual reciprocity boundary element method has been presented for the solution of laminar heat convection problem in a concentric annulus imposed with constant heat flux. DRBEM matrix is formulated to perform the numerical implementation, and five cases of boundary element discretization are tested with the corresponding number of internal points. Five radial locations are selected to obtain the velocity and temperature solutions. Test results are shown to be in excellent agreement with exact solutions for the above 36 element case.

REFERENCES

[1] C., Brătianu, Metode cu elemente finite în dinamica fluidelor, Editura Academiei, Bucureşti, 1983.

[2] P. K., Banerjee, The Boundary Element Method in Engineering, McGraw-Hill, London, 1992.

[3] A., Bejan, Heat Transfer, John Wiley & Sons, New York , 1993.

[4] C. A., Brebbia, Boundary Element Techniques, Springer-Verlag, Berlin, 1984.

[5] I., Caius , D., Homentcovschi, N., Marcov, A., Ncolau, Matematici clasice şi moderne, Editura Tehnică, Bucureşti, 1983.

[6] A. J., Davies, The Finite Element Method, Clarendon Press, Oxford, 1980.

[7] D., Gârbea , Analiza cu elemente finite, Editura Tehnică, Bucureşti, 1990.

[8] V., Olariu, C., Brătianu , Modelarea numerică cu elemente finite, Editura Tehnică, Bucureşti, 1986.

[9] J. N., Reddy, An introduction to the Finite Element Method, Secand Edition, McGraw-Hill Book Company, New York, 1993.

[10] J. N., Reddy, G. K., Garling, The Finite Element Method in Heat Transfer and Fluid Dynamics, Secand Edition, CRT Press, 2000.

[11] G., Strang, G., Fix, An analysis of the Finite Element Method, Prentice -Hall, New Jersey, 1975.

[12] D., Ştefănescu ş.a. , Transfer de căldură şi masă. Teorie şi aplicaţii, Editura Didactică şi Pedagogică, Bucureşti, 1983.

[13] P., Titus , C. I., Gheorghiu, Metode element finit şi aplicaţii, Editura Academiei RSR, Bucureşti, 1987.

[14] G., Vraciu, A., Popa, Metode numerice cu aplicaţii în tehnica de calcul, Editura Scrisul Românes, Craiova, 1982.

[15] O. C., Zienkiewicz, Finite Element Method, McGraw-Hill, London, 1977.

[16] O. C., Zienkiewicz, R. L., Taylor, Tthe Finite Element Method, Vol. 1, The Basis. Fifth Edition, Butterworth and Heinemann, London, 2000.

[17] L., Landau, E., Lifchitz, Physique théoretique, Tome 6, Mécanique des fluides, Deuxième édition revue et complétée, Edition MIR Moscou, 1989.