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NUMERICAL MODELING OF TEMPERATURE FILDS IN STADY-STATE HEAT TRANSFER PROBLEMS USING BOUNDARY ELEMENT METHOD

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Abstract: In this paper the basic ideas of numerical analysis with boundary (constant) elements of conductive thermal fields generated or induced into plain walls in steady state regime are developed. The temperature distribution in two variants of a metallic plaque is analyzed using boundary element method, implemented in software developed by the author and analytical method.

Keywords: Heat conduction, Steady state regime, Boundary elements, Mathematical model, Computer program.

1. INTRODUCTION

Modern computational techniques facilitate solving problems with imposed boundary conditions using different numerical methods [1–4]. Numerical analysis of heat transfer has been independently though not exclusively, developed in following main streams: the finite differences method [5], the finite element method [6] and boundary element method [7], [8].

The finite differences method (FDM) is based on the differential equation of the heat conduction, which is transformed into a numerical one. The temperature values will be calculated in the nodes of the network. Using this method convergence and stability problem can appear. The finite element method (FEM) is based on the integral equation of the heat conduction. This is obtained from the differential equation using variational calculus.

The temperature distribution is analyzed in a solid body, with linear variation of the properties, using a software realized by the author on basis of the BEM.

2. ANALITICAL MODEL OF HEAT CONDUCTION

The temperature in a solid body is a function of the time and space coordinates. The points corresponding to the same temperature value belong to an isothermal surface. This surface in a two-dimensional Cartesian system is transformed into an isothermal curve.

The heat flow rate Q represents the heat quantity through an isothermal surface S in the time unit:

$$Q = \int_S q \, \mathrm{d}s \tag{1}$$

where the density of heat flow rate q is given by the Fourier law:

$$q = -\lambda \frac{\partial t}{\partial n} = -\lambda \, \mathrm{grad}\, t \tag{2}$$

in which λ is the thermal conductivity of the material.

The thermal conductivity of the building materials is the function of the temperature and variation can accordingly be expressed as:

$$\lambda = \lambda_0 \left[1 + b(t - t_0)\right] \tag{3}$$

in which: λ_0 is the thermal conductivity corresponding to the t_0 temperature; b – material constant.

If there is heat conduction within an inhomogeneous and anisotropy material, considering the heat conductivity constant in time, the temperature variation in space and time is given by the Fourier equation:

$$\rho c \frac{\partial t}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial t}{\partial z}\right) + Q_0 \tag{4}$$

in which: t is the temperature; τ – time; ρ – material density; c – specific heat of the material; λ_x, λ_y, λ_z – thermal conductivity in the directions x, y and z; Q_0 – power of the internal sources.

To solve the differential equations it is necessary to have supplementary equations. These equations contain the geometrical conditions of the analysis field, the starting conditions (at $\tau = 0$) and the boundary conditions. The boundary conditions (Fig. 1) describe the interaction between the analyzed field and the surroundings. In function of these interactions different conditions are possible:

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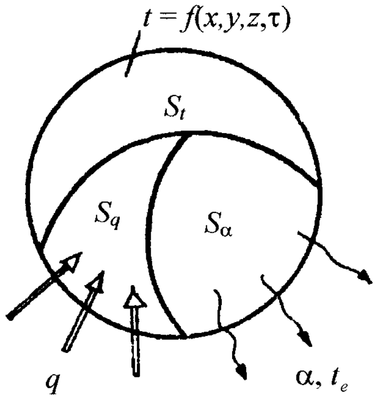

Figure 1. Boundary conditions

– the *Dirichlet* (type I) boundary conditions give us the temperature values on the boundary surface S_t of the analyzed field like a space function constant or variable in time:

$$t = f\left(x, y, z, \tau\right) \qquad (5)$$

– the *Neumann* (type II) boundary conditions gives us the value of the density of heat flow rate through the S_q boundary surface of the analyzed field:

$$q = \lambda_x \frac{\partial t}{\partial x} n_x + \lambda_y \frac{\partial t}{\partial y} n_y + \lambda_z \frac{\partial t}{\partial z} n_z \qquad (6)$$

in which: n_x, n_y, n_z are the cosine directors corresponding to the normal direction on the S_q boundary surface.

– the *Cauchy* (type III) boundary conditions gives us the external temperature value and the convective heat transfer coefficient value between the S_α boundary surface of the body and the surrounding fluid:

$$\alpha\left(t - t_e\right) = \lambda_x \frac{\partial t}{\partial x} n_x + \lambda_y \frac{\partial t}{\partial y} n_y + \lambda_z \frac{\partial t}{\partial z} n_z \, (7)$$

in which: α is yhe convective heat transfer coefficient from S_α to the fluid (or inversely); t_e – the fluid temperature.

The analytical model described by the equations (4)…(7) can be completed with the material equations which provide us information about variation of the material properties depending on temperature. In the case of material with linear physical properties, this equations (λ = const.) are not used in the model.

3. NUMERICAL MODEL WITH BOUNDARY ELEMEMTS OF HEAT CONDCTIVITY IN STEADY STATE REGIME

Although thermal phenomena take place in thru dimensional bodies, the thermal fields that occur have predominant variations in certain directions. This is why the analysis of thermal field in plain or cylindrical walls is usually performed using two dimensional computational models.

In steady state heat transfer the temperature is a constant of time, and for two dimensional problems the temperature does not vary in the direction of axis Oz.

In the case of a flat wall, inside the analysis field, the heat conductivity in steady state regime is modelled by the Laplace equation [8]:

$$\nabla^2 t = 0 \qquad (8)$$

On Γ_t portion of boundary Γ of the analysis field Dirichlet boundary conditions are imposed and left corner portion Γ_q Neumann boundary conditions are imposed.

In order to determine the temperature on the boundary of the analysis field one uses the following integral equation [7], [8]:

$$c(\zeta)t(\zeta) + \int_\Gamma t(\overset{o}{X}) v^*(\zeta, \overset{o}{X}) \mathrm{d}\Gamma(\overset{o}{X} = \int_\Gamma \frac{\partial t(\overset{o}{X})}{\partial n} u^*(\zeta, \overset{o}{X}) \mathrm{d}\Gamma(\overset{o}{X}) \qquad (9)$$

where: ζ is the point in which one writes the integral equation (source point); $c(\zeta)$ – a coefficient; $\overset{o}{X}$ – the current integration point; $u^*(\zeta, \overset{o}{X}) = \frac{1}{2\pi} \ln \frac{1}{r(\zeta, \overset{o}{X})}$ – fundamental solu-tion; $v^* = \frac{\partial u^*}{\partial n}$ – normal derivative of this solution.

The distance $r(\zeta, \overset{o}{X})$ between the current point $\overset{o}{X}$ and the source point ζ is calculated with the relation:

$$r(\zeta, \overset{o}{X}) = \left\{ \left[x(\overset{o}{X}) - x(\zeta) \right]^2 + \left[y(\overset{o}{X}) - y(\zeta) \right]^2 \right\}^{\frac{1}{2}} \qquad 10)$$

Boundary Γ is discretised into N constant boundary elements for which one considers temperatures t_j, respectively the normal derivative $(\partial t/\partial n)_j$ constant and equal to the mid point (node) value of the element. Thus the integral equation is obtained under the following discretised form:

$$c_i t_i + \sum_{j=1}^{N} t_j \int_{\Gamma_i} v^*(\zeta, \overset{o}{X}) \, d\Gamma(\overset{o}{X}) = \sum_{j=1}^{N} \left(\frac{\partial t}{\partial n} \right)_j \int_{\Gamma_l} u^*(\zeta, \overset{o}{X}) \, d\Gamma(\overset{o}{X}) \qquad (11)$$

or

$$c_i t_i + \sum_{j=1}^{N} \hat{A}_{ij} t_j = \sum_{j=1}^{N} B_{ij} \left(\frac{\partial t}{\mathrm{d}n} \right)_j \qquad (12)$$

in which coefficients $\hat{A}_{ij}$ and B_{ij} have the expressions:

$$\hat{A}_{ij} = \int_{\Gamma_j} v^*(\zeta, \overset{o}{X})\, d\Gamma(\overset{o}{X});\ B_{ij} = \int_{\Gamma_j} u^*(\zeta, \overset{o}{X})\, d\Gamma(\overset{o}{X}) \quad i \neq j \tag{13}$$

When $i = j$ these become:

$$A_{ii} = \frac{1}{2} + \hat{A}_{ii}; \quad B_{ii} = \frac{l_i}{2\pi}\left(1 - \ln\frac{l_i}{2}\right) \tag{14}$$

Explicitely, equation (12) generates a liniar and compatible system of N equations with $2N$ unknowns [t_j and $(\partial t/\partial n)_j$] and after implementing the boundary conditions, the number of unknowns is reduced to N. In the case of constant boundary elemnents, coefficient c_i has the value 1/2. Coefficients $\hat{A}_{ij}$ and B_{ij} from (13) is computed using a Gauss quadrature [8], [9]:

$$\hat{A}_{ij} = \frac{l_j}{2}\sum_{k=1}^{m} v_k^* w_k; \quad B_{ij} = \frac{l_j}{2}\sum_{k=1}^{m} u_k^* w_k \tag{15}$$

in which l_j is the length of the j boundary element.

Introducing notations: n_x=cos(n,x); n_y = cos(n, y) and using, for $\forall \overset{o}{X} \in \Gamma$, the parametric equations:

$$x = A\xi + B; \quad y = C\xi + D, \quad \xi \in [-1,1] \tag{16}$$

where: $x \in [x_j, x_{j+1}]$ and $y \in [y_j, y_{j+1}]$, the following relations are obtained:

$$n_x = \frac{-C}{\sqrt{A^2 + C^2}}; \quad n_y = \frac{A}{\sqrt{A^2 + C^2}} \tag{17}$$

in which (x_j, y_j) and (x_{j+1}, y_{j+1}) are the extremities of the boundary element j.

The analysis field is transformed into a dimensionless one by replacing the dimensional variables (x, y) with dimensionless ones (x^*, y^*):

$$x^* = \frac{x}{x_{\max}}; \quad y^* = \frac{y}{x_{\max}} \tag{18}$$

in which $x_{\max}$ is the maximum extension of the analysis field after axis Ox.

In order to determine the temperature inside of the analysis field is used the integral representation:

$$t_i(\zeta_i) = \int_{\Gamma} \frac{\partial t(\overset{o}{X})}{\partial n^*} u^*(\zeta_i, \overset{o}{X})\, d\Gamma(\overset{o}{X}) - \int_{\Gamma} \frac{\partial t(\overset{o}{X})}{\partial n^*} v^*(\zeta_i, \overset{o}{X})\, d\Gamma(\overset{o}{X}) \tag{19}$$

in which: $\zeta_i \in \overset{o}{\Omega}$, where $\overset{o}{\Omega}$ represent the inside of the analysis field Ω ($\Omega = \overset{o}{\Omega} \cup \Gamma$).

After the discretization of boundary Γ into N constant boundary elements one obtains the integral equation under discretized form:

$$t_i = \sum_{j=1}^{N}\left(\frac{\partial t}{\partial n^*}\right)_j \int_{\Gamma_j} u^*(\zeta_i, \overset{o}{X})\, d\Gamma(\overset{o}{X}) - \sum_{j=1}^{N} t_j \int_{\Gamma_j} v^*(\zeta_i, \overset{o}{X})\, d\Gamma(\overset{o}{X}) \tag{20}$$

which can be writhen as such:

$$t_i = \sum_{j=1}^{N} \overline{B}_{ij}\left(\frac{\partial t}{\partial n^*}\right)_j - \sum_{j=1}^{N} \overline{A}_{ij} t_j \tag{21}$$

Coefficients $\overline{A}_{ij}$ and $\overline{B}_{ij}$ are evaluated using a Gauss quadrature:

$$\overline{A}_{ij} = \frac{l_j}{2}\sum_{k=1}^{m} v_k^* w_k; \quad \overline{B}_{ij} = \frac{l_j}{2}\sum_{k=1}^{m} u_k^* w_k \tag{22}$$

in which: m is the number of Gauss type points; w_k – weight coefficients.

Temperatures t_i from points ζ_i are easily determined taking into account that values t_j and $(\partial t/\partial n^*)_j$ are known on the analysis field boundary, and coefficients $\overline{A}_{ij}$ and $\overline{B}_{ij}$ are computed with equation (16).

By knowing values t_j and t_i of the temperature on the analysis field boundary, the group of coordinate points (x^*, y^*) for which t = const. represents the isothermal curves. The numerical model develobed above, based on BEM, was implemented by the author in programs TEMPBEM and TERMINBEM elaborated in FORTRAN programming language, for IBM–PC compatible computers.

4. NUMERICAL APPLICATION

In figures 2 and 3 are considered two variants of a metallic plaque, with dimensions 40×40×70 mm, for which one determines the temperature field using BEM and analytical method (ANM). In figures 4 and 5 are presented the dimensionless analysis domains together with mixed boundary conditions for these boundaries.

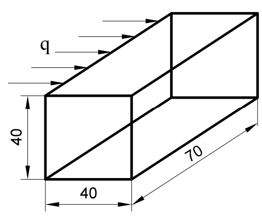

Figure 2. Metallic plaque

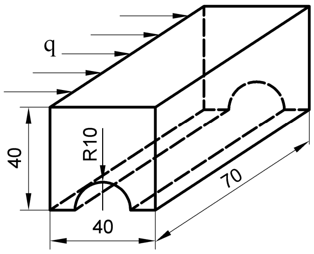

Figure 3. Metallic plaque with a semicylindrical cut off

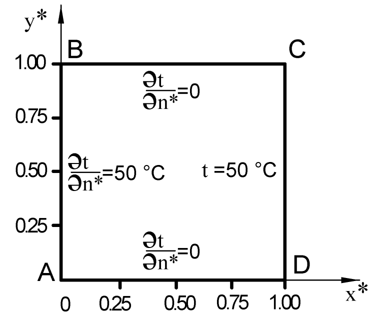

Figure 4. Boundary conditions for metallic plaque

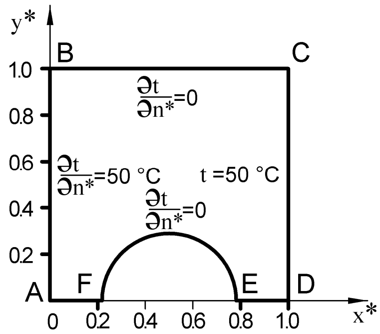

Figure 5. Boundary conditions for metallic plaque with semi cylindrical cut–out

For metallic plaque in figure 2 the boundary can be discretized into N = 16 boundary elements, one states 9 internal points (Figure 6) and one applies the computational model based on BEM. The numerical results obtained by means of an IBM computer are presented in Table 1, comparatively with the ones obtained with ANM [5].

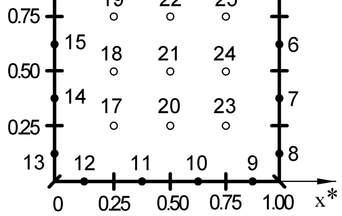

Figure 6. Discretization of the boundary and internal points of analysis field

Table1a. Coordinate values

Point	Coordinates	
j	x_j^*	y_j^*
1	0.125	1.000
2	0.375	1.000
3	0.625	1.000
4	0.875	1.000
5	1.000	0.875
6	1.000	0.625
7	1.000	0.375
8	1.000	0.125
9	0.875	0.000
10	0.625	0.000
11	0.375	0.000
12	0.125	0.000
13	0.000	0.125
14	0.000	0.375
15	0.000	0.625
16	0.000	0.875
17	0.250	0.250
18	0.250	0.500
19	0.250	0.750
20	0.500	0.250
21	0.500	0.500
22	0.500	0.750
23	0.750	0.250
24	0.750	0.500
25	0.750	0.750

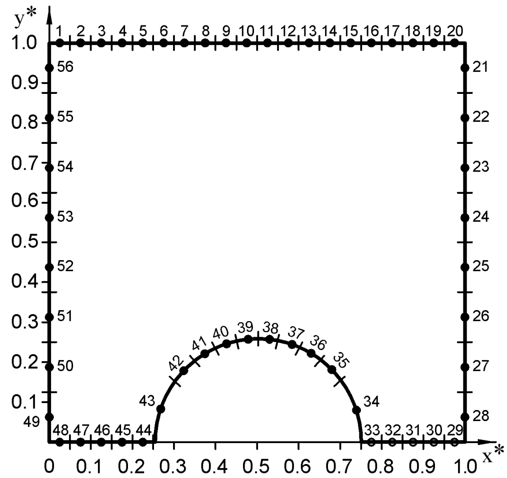

Figure 7. Boundary discretization for the plaque with semi cylindrical cut–out

The absolute percentage value of the relative difference toward the analytical solution, for both the temperature ε_t and it normal derivative ε_{ndt} is defined by:

$$\varepsilon_t = \left|(t_{ANM} - t_{BEM})/t_{ANM}\right|\cdot 100;$$

$$\varepsilon_{ndt} = \left|\left[\left(\frac{\partial t}{\partial n^*}\right)_{ANM} - \left(\frac{\partial t}{\partial n^*}\right)_{BEM}\right]\left(\frac{\partial t}{\partial n^*}\right)^{-1}_{ANM}\right|\cdot 100 \qquad (23)$$

Table1b. The values t_j and $\left(\frac{\partial t}{\partial n^*}\right)_j$

BEM		ANM	
t_j	$\left(\frac{\partial t}{\partial n^*}\right)_j$	t_j	$\left(\frac{\partial t}{\partial n^*}\right)_j$
93.052	0.000	93.75	0.0
80.705	0.000	81.25	0.0
68.299	0.000	68.75	0.0
55.821	0.000	56.25	0.0
50.000	-51.704	50.00	-50.0
50.000	-48.290	50.00	-50.0
50.000	-48.290	50.00	-50.0
50.000	-51.704	50.00	-50.0
55.821	0.000	56.25	0.0
68.299	0.000	68.75	0.0
80.705	0.000	81.25	0.0
93.052	0.000	93.75	0.0
98.776	50.000	100.0	50.0
99.308	50.000	100.0	50.0
99.308	50.000	100.0	50.0
98.776	50.000	100.0	50.0
86.836	0.000	87.50	0.0
86.876	0.000	87.50	0.0
86.836	0.000	87.50	0.0
74.521	0.000	75.00	0.0
74.536	0.000	75.00	0.0
74.521	0.000	75.00	0.0
62.205	0.000	62.50	0.0
62.240	0.000	62.50	0.0
62.205	0.000	62.50	0.0

Taking into account the results from Table 1 when applying equations (23), acceptable values have been obtained for ε_t and ε_{ndt} ($\varepsilon_t < 1.3\%$, $\varepsilon_{ndt} < 3.5\%$) even if the number of boundary elements considered is small.

For metallic plaque in figure 3 the boundary can be discretized into N = 56 constant boundary elements (Figure 7) and using BEM was determined isothermal curves presented in Figure 8.

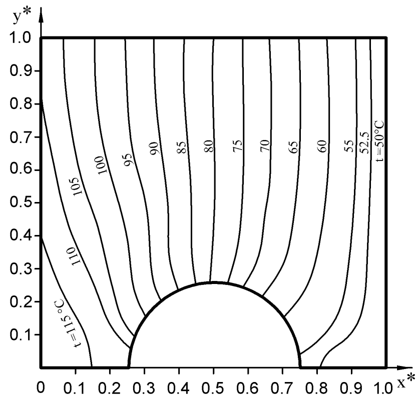

Figure 8. Temperature distribution for the plaque with semi cylindrical cut–out

5. CONCLUSIONS

The numerical computation of the temperature field, on the basis of the boundary element method, has led to close values to the ones determined analytically even if a small number of boundary elements and respectively internal points of the analysis domain was used. Using the presented method, different simulation programms could be realized what makes it possible to effectuate a lot of different numerical experiments of practical problems.

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