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NUMERICAL SIMULATION OF TEMPERATURE FIELD IN STADY-STATE AND TRANSIENT REGIME FOR TWO-DIMENSIONAL PROBLEMS USING FINITE ELEMENT METHOD

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Abstract: **This paper presents the modality to solve heat conduction steady-state and transient regime in the case of two-dimensional problems, using the Finite Element Method. Firstly the paper develops the analytical model with linear triangular elements in case of thermal conduction in steady-state and transient regimes, and then presents the numerical examples with corresponding results.**

Keywords: finite element, head conduction, steady-state regime, transient regime.

1. INTRODUCTION

Transmission of heat is a natural process of energy transfer from bodies with higher temperature to those with lower temperature and furthernore inside a body from areas with higher temperatures to those of a lower temperature. Heat is transmitted by conduction , convection and radiation. Thermal conduction takes place in three-dimensional bodies and their resulting thermal fields which manifest themselves in a certain way after the coordinate axes. There are such bodies as plain walls, bars, wires for which two-dimensional and one-dimensional models are used to analyze thermal fields.

Development of electronic computers has allowed solving various problems with imposed conditions, if one appeals to different numerical methods developed in modern literature. From this point of view thermal conduction is a wholesome domain of applicability of the Finite Element Method, which constitutes the model in full for the studied phenomenon.

2. DIFFERENTIAL EQUATION OF HEAT CONDUCTION

Temperature θ in a body depends on spatial coordinates x, y, z and time t as such:

$$\theta = f(x, y, z, t) \qquad (1)$$

If $\frac{\partial \theta}{\partial t} = 0$, it means that we have a steady temperature field and if $\frac{\partial \theta}{\partial t} \neq 0$ it is a transient temperature field. Depending on the number of coordinates by which the temperature varies, the field is called one-dimensional, two-dimensional or three-dimensional. The set of points within a body with constant temperature forms an isothermal surface and in the case of two-dimensional fields an isothermal curve occurs. The amount of heat that passes through a surface, measured by a time unit, is called heat flow and is defined by the following:

$$Q = \int_S q \, dS \qquad (2)$$

in which q is the heat flow or unitary heat flow which according to Fourier's law is expressed as such [12]:

$$q = -\lambda \frac{\partial \theta}{\partial n} = -\lambda \, \vec{n} \cdot \nabla \theta \qquad (3)$$

In equation (3) λ was used to note the thermal conductivity coefficient and n notes the normal axis to surface S. Variation of temperature in time and space, inside the body or a fluid environment, is given by Fourier's equation [12]:

$$\frac{\partial(\rho c_p \theta)}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_x \frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial \theta}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial \theta}{\partial z}\right) + q_V \qquad (4)$$

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, in which: ρ - specific body mass; c_p - specific heat; q_v - unitary volumetric heat flow; λ_x, λ_y, λ_z - thermal coductivity coefficients according to axes x,y,z.

For solving equation (4) we must impose: a) - initial condition for transient regimes and b) - spatial conditions [7].

a) The time limit conditions establish the temperature distribution at the beginning of the process, as such:

$$\theta = f(x,y,z,t=0) = f(x,y,z) \qquad (5)$$

b1) The primary space limit conditions specify that the temperature is imposed on S_θ surface, as such:

$$\theta_{S_\theta} = f(x,y,z,t) \qquad (6)$$

b2) Second type limit conditions mention that the thermal flow is imposed on the body's surface S_q thus:

$$q = \lambda_x \frac{\partial\theta}{\partial x} n_x + \lambda_y \frac{\partial\theta}{\partial y} n_y + \lambda_z \frac{\partial\theta}{\partial z} n_z \qquad (7)$$

in which , n_x, n_y, n_z are notations for the directing cosines of the normal to the surface.

b3) Third type spatial limit conditions state that heat transfer occurs through convection on surface S_α with the known parameters, as such:

$$\alpha\left(\theta - \theta_E\right) = \lambda_x \frac{\partial\theta}{\partial x} n_x + \lambda_y \frac{\partial\theta}{\partial y} n_y + \lambda_z \frac{\partial\theta}{\partial z} n_z \qquad (8)$$

in which: α - is the convection cofficient from surface S_α to the environment or viceversa and θ_E - is the outside temperature.

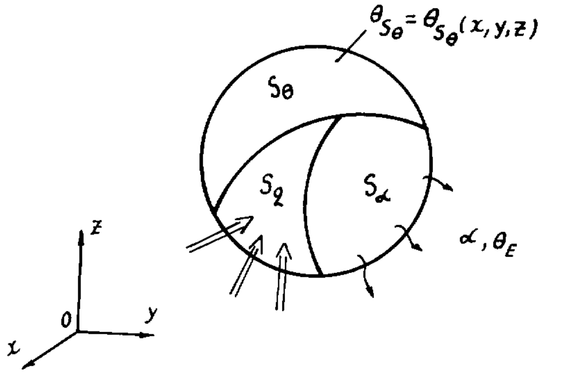

Fig. Space limit conditions

3. MATHEMATICAL MODEL WITH FINITE ELEMENTS IN SOLVING TWO-DIMENSIONAL CONDUCTIVE HEAT TRANSFER IN A STEADY–STATE REGIME

The paper addresses the issue of flat two-dimensional problems which imply that the temperature does not vary in the direction of the z axis, thus $\frac{\partial\theta}{\partial z} = 0$. We will consider the steady heat transfer which means $\frac{\partial\theta}{\partial t} = 0$. If we customize the differential equation (4) and the limit boundary conditions (5), (6), (7) for two-dimensional problems and we use variation formula to obtain analogous finite element equations, we get the expression of functional J [6], [7], [10].

$$J = \int_V \left\{ \frac{1}{2}\left[\lambda_x \left(\frac{\partial\theta}{\partial x}\right)^2 + \lambda_y \left(\frac{\partial\theta}{\partial y}\right)^2 \right] - q_V\theta \right\} dV - \int_{S_q} q\theta\, dS + \int_{S_\alpha} \alpha\theta\left(\frac{1}{2}\theta - \theta_E\right) dS \qquad (9)$$

Dividing the analysis domain into m finite elements, leads to summing every finite element of functional J_e, thus having:

$$J = \sum_{e=1}^{m} \int_{V_e} \frac{1}{2}\left[\lambda_x \left(\frac{\partial\theta_e}{\partial x}\right)^2 + \lambda_y \left(\frac{\partial\theta_e}{\partial y}\right)^2 \right] dV - \int_{V_e} q_V\theta_e\, dV - \int_{S_{q\,e}} q\theta_e\, dS + \int_{S_{\alpha\, e}} \alpha\theta_e \left(\frac{1}{2}\theta_e - \theta_E\right) dS \qquad (10)$$

where index „e” refers to a random finite element. Temperature θ_e in any point of the finite element is expressed on the basis of nodal temperatures and the form functions using [8],[16]:

$$\theta_e = \left[N_1 \quad N_2 ....N_n\right] \begin{Bmatrix} \theta_1 \\ \theta_2 \\ . \\ \theta_n \end{Bmatrix} = \left[N\right]\left\{\theta\right\}_e \qquad (11)$$

where n is the number of nodes in the finite element.

In the case of functional J, we have the partial derivatives of temperature θ_e in relation to variables x and y which can be written under matrix form:

$$\begin{Bmatrix} \dfrac{\partial \theta_e}{\partial x} \\ \dfrac{\partial \theta_e}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} \cdots\cdots & \dfrac{\partial N_n}{\partial x} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} \cdots\cdots & \dfrac{\partial N_n}{\partial y} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ . \\ \theta_n \end{Bmatrix} = \qquad (12)$$

$$= [B]\{\theta\}_e$$

Matrix $[B]$ contains form functions' derivatives in relation to coordinates x and y. If we note the matrix of thermal conductivity coefficients witth:

$$[D] = \begin{bmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{bmatrix} \qquad (13)$$

and taking into account relations:

$$\begin{gathered} ([B]\{\theta\}_e)^T = \{\theta\}_e^T [B]^T \\ ([N]\{\theta\}_e)^2 = ([N]\{\theta\}_e)^T ([N]\{\theta\}_e) = \\ \{\theta\}_e^T [N]^T [N]\{\theta\}_e \end{gathered} \qquad (14)$$

and also that $dV = h\,dA$ and $dV = h\,dl$, where h is the finite element's thickness ; dA - differential area element and dl - differential length element, we obtain the formula for minimizing functional J_e:

$$\begin{gathered} \frac{\partial J_e}{\partial \{\theta\}_e} = h \begin{pmatrix} \int_{A_e} [B]^T [D][B] dA + \\ + \int_{l_e} \alpha [N]^T [N] dl \end{pmatrix} \{\theta\}_e \\ - h\int_{A_e} q_V [N]^T dA - h\int_{l_e} q[N]^T dl - \\ \int_{l_e} \alpha\theta_E [N]^T dl \end{gathered} \qquad (15)$$

We consider the finite element's thickness h as constant and equal to the unit used in plain problems. Under condensed form relation (15) becomes:

$$\frac{\partial J_e}{\partial \{\theta\}_e} = [k]\{\theta\}_e - \{f\} \qquad (16)$$

In equation (16) matrix $[k]$ of the finite element in which occur physical processes of heat transfer through conductivity inside the element and through convection on the boundary has the following expression [7],[3]:

$$[k] = h \begin{pmatrix} \int_{A_e} [B]^T [D][B] dA + \\ + \int_{l_{3e}} \alpha [N]^T [N] dl \end{pmatrix} = [k_1] + [k_2] \qquad (17)$$

and

$$\begin{gathered} \{f\} = h\int_{A_e} q_v [N]^T dA + h\int_{l_{2e}} q[N]^T dl + \\ h\int_{l_{3e}} \alpha\theta_E [N]^T dl = \{f_{q_V}\} + \{f_q\} + \{f_\alpha\} \end{gathered} \qquad (18)$$

, is the heat load vector. If we take into account (10), (15), (16) we obtain the global equations system:

$$[K]\{\theta\} = \{F\} \qquad (19)$$

in which $[K] = \sum_{e=1}^{m} [k]$ is the global conuctivity matrix, $\{F\} = \sum_{e=1}^{m} \{f\}$ - global heat load vector, $\{\theta\}$ - global temperature vector.

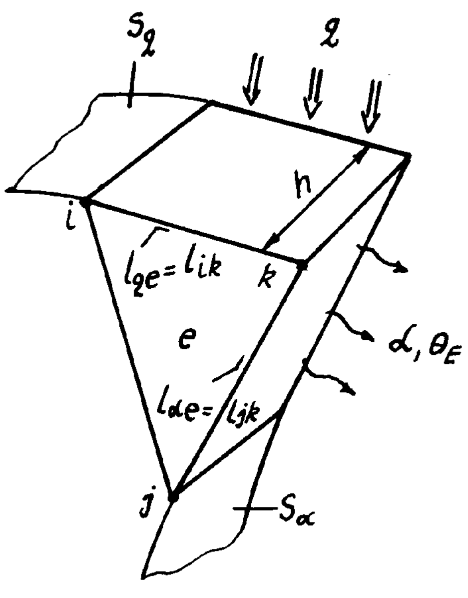

Figure 2. The triangular finite element with heat exchange by thermal flow and convection.

4. EXPRESSION OF CONDUCTIVITY MATRIX AND THERMAL LOAD VECTOR ON THE TRIANGULAR FINITE ELEMENT

In the case of a triangular finite element with three nodes i, j, k, temperature θ_e in any point of the element is written on the basis of nodal values and form functions:

$$\theta_e = N_i\theta_i + N_j\theta_j + N_k\theta_k \tag{20}$$

Matrix $[k_1]$ and $[k_2]$, in conformity with [7] have the expression:

$$[k_1] = \frac{h}{4A_e}\begin{bmatrix} \lambda_x b_i b_i + \lambda_y c_i c_i & \lambda_x b_i b_j + \\ \lambda_y c_i c_j & \lambda_x b_i b_k + \lambda_y c_i c_k \\ \lambda_x b_j b_i + \lambda_y c_j c_i & \lambda_x b_j b_j + \\ \lambda_y c_j c_j & \lambda_x b_j b_k + \lambda_y c_j c_k \\ \lambda_x b_k b_i + \lambda_y c_k c_i & \lambda_x b_k b_j + \\ \lambda_y c_k c_j & \lambda_x b_k b_k + \lambda_y c_k c_k \end{bmatrix} \tag{21}$$

$$[k_2] = h\alpha\int_{l_{3e}}\begin{bmatrix} N_iN_i & N_iN_j & N_iN_k \\ N_jN_i & N_jN_j & N_jN_k \\ N_kN_i & N_kN_j & N_kN_k \end{bmatrix} dl \tag{22}$$

If we introduce length coordinates [7],[8],[15] and convection exchange occurs on side jk, ij, ki and we take into account same index products j or k, i or j, k or i, respectively j and k, i and j, k and i all different, we obtain:

$$[k_2] = \frac{h\alpha\, l_{3e}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[k_2] = \frac{h\alpha\, l_{3e}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{23}$$

$$[k_2] = \frac{h\alpha\, l_{3e}}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Considering internal sources' flow as constant on the finite element and the heat exchange through imposed thermal flow through side ki of length $l_{ki} = l_{2e}$ and through convection on side jk of length $l_{jk} = l_{3e}$, we have as a result[7]:

$$\{f_{q_V}\} = \frac{hq_V A_e}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\{f_q\} = \frac{hql_{ki}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \tag{24}$$

$$\{f_\alpha\} = \frac{h\alpha\theta_E}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

5. FINITE ELEMENT MODEL IN TRANSIENT REGIME

Taking into account the heat transfer equation (4) and the functional (9), we notice that for heat transfer in a transient regime, instead of $-\int_V q_V\theta\, dV$ from the steady regime, we also have the expression:

$$\begin{aligned} &-\int_V\left(q_V - \rho\, c_p\,\frac{\partial\theta}{\partial t}\right)\theta\, dV = \\ &-\int_V q_V\theta\, dV + \int_V \rho\, c_p\,\frac{\partial\theta}{\partial t}\,\theta\, dV \end{aligned} \tag{25}$$

This imposes a new term to consider J_e^* for functional J_e on the finite element thus:

$$J_e^* = \int_V (\rho\, c_p)_e\,\frac{\partial\theta_e}{\partial t}\,\theta_e\, dV \tag{26}$$

which needs minimizing. On the finite element we obtain the following linear differential equation system in relation to time:

$$[C]\frac{\partial\{\theta\}_e}{\partial t} + [k]\{\theta\}_e - \{f\} = 0 \tag{27}$$

in which the calorific capacity matrix $[C]$ for the triangular finite element has the expression:

$$[C] = \frac{c_p\rho A_e}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{28}$$

In order to integrate in time a differential equation system (27) and to obtain the temperatures field, we appeal to the weighted residuals method and then we have a reccurence formula:

$$\left(\left[C\right]+\frac{2\Delta t}{3}\left[k\right]\right)\left\{\theta\right\}_{e\,j}=$$
$$\left(\left[C\right]-\frac{\Delta t}{3}\left[k\right]\right)\left\{\theta\right\}_{e\,i}+\Delta t\left\{f\right\}_{j} \qquad (29)$$

The reccurence relation (29) allows temperatures $\left\{\theta\right\}_{e\,j}$ determination in the finite element's nodes at moment j, in relation to temperatures $\left\{\theta\right\}_{e\,j}$ at previous moment i.

$$\left[C\right]+\frac{2\Delta t}{3}\left[k\right]=\left[k\right]$$
$$\left[C\right]-\frac{\Delta t}{3}\left[k\right]=\left[p\right] \qquad (30)$$
$$\left[p\right]\left\{\theta\right\}_{ei}+\Delta t\left\{f\right\}_{j}=\left\{f\right\}_{t}$$

we have the finite element equation written in the form of :

$$\left[k\right]_{t}\left\{\theta\right\}_{e}=\left\{f\right\}_{t} \qquad (31)$$

The system of equations (31) is determined for all finite elements and then is assembled for obtaining the global system which allows determining temperatures in global nodes at time step j in relation to previous moment i.

6. NUMERICAL RESULTS FOR HEAT TRANSFER IN STEADY-STATE REGIME

We consider the analysis domain a rectangle having dimensions $60\,cm$ and $40\,cm$ and is divided into 48 triangular finite elements and 35 nodes.This together with boundary conditions are presented in figure 3. Numerical computing is done with the following values: $q_x = 2326\ W/m^2$ and $q_y = 930\ W/m^2$ - unitary thermal flow in relation to axis coordinates; $\alpha_x = \alpha_y = 23\ W/m^2\ {}^\circ C$ - convection coefficients and $\theta_{Ex} = \theta_{Ey} = 10\ {}^\circ C$ is outside temperature value, $\lambda_x = 12\ W/m^2\ {}^\circ C$, $\lambda_y = 6\ W/m^2\ {}^\circ C$ conductivity coefficients. Numerical results are obtained on the basis of program TCSTAMEF in FORTRAN programming language.

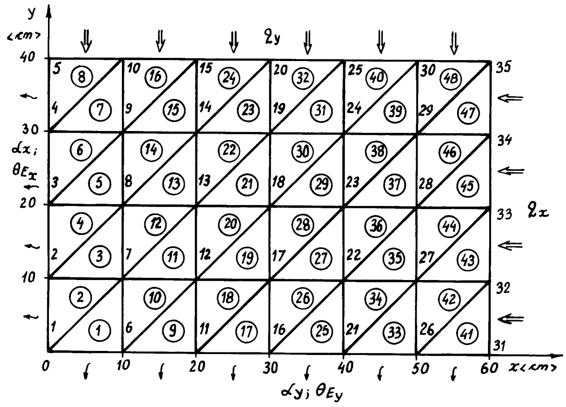

Figure 3.Analysis domain and boundary conditions.

Table 1.Nodes coordinates and temperature values

Nod	x cm	y cm	θ °C	Nod	x cm	y cm	θ °C
1	0	0	46.96	19	30	30	133.05
2	0	10	61.98	20	30	40	149.25
3	0	20	75.46	21	40	0	82.43
4	0	30	88.31	22	40	10	108.24
5	0	40	102.16	23	40	20	130.17
6	10	0	55.10	24	40	30	149.08
7	10	10	72.28	25	40	40	165.64
8	10	20	88.14	26	50	0	94.65
9	10	30	103.10	27	50	10	123.35
10	10	40	118.11	28	50	20	146.68
11	20	0	63.38	29	50	30	166.29
12	20	10	83.22	30	50	40	183.07
13	20	20	101.29	31	60	0	110.87
14	20	30	117.86	32	60	10	141.14
15	20	40	133.58	33	60	20	165.06
16	30	0	72.30	34	60	30	184.92
17	30	10	95.07	35	60	40	201.78
18	30	20	115.17				

7. NUMERICAL RESULTS IN CASE OF CONDUCTIVE HEAT TRANSFER IN TRANSIENT REGIME

For numerically solving the temperature field in the case of heat transfer in transient regime we will use the analysis domain presented in fig, 4. The body has internal heat sources uniformly distributed and is adiabatic insulated towards the environment.

Initial temperature is $\theta_0 = 10\,^\circ C$; density value is $\rho = 7650\,Kg/m^3$; $c_p = 492.6\,J/Kg\,^\circ C$ - specific heat; unitary volumetric heat flow, $q_v = 4.652\times10^5\,W/m^3$; conductivity coefficient $\lambda_x = \lambda_y = 46.5 W/m\,^\circ C$. We will choose a time interval $\Delta t = 36\,s$ and 10 time steps. Numerical results were obtained with TCIRTMEF computer program.

Due to the material's homogenity, the uniform distributions of heat sources and adiabatic insulation of the body, at a given time step we obtain the same temperature in all the nodes. The temperature values corresponding to time steps are presented in table 2, and temperature graph in relation to time is given fig.5

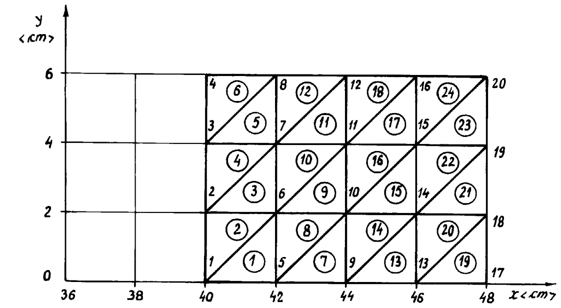

Figure 4. Analysis domain.

Table 2.Temperature θ values at time t

$\theta <^\circ C>$	14.4	18.8	23.3	27.7	32.2
$t < s >$	36	72	108	144	180
$\theta <^\circ C>$	36.6	41.1	45.5	49,9	54.4
$t < s >$	216	252	288	324	360

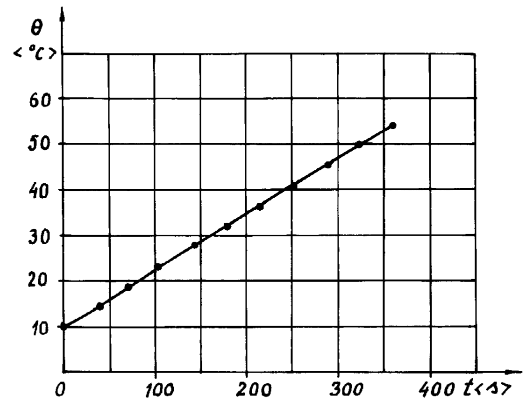

Figure5. Temperature variation with time

8. CONCLUSIONS

Numerical modelling with finite elements represents an effective mathematical instrument for the accurately determining conductive thermal fields.

With the help of program TCSTAMEF we determined temperatures in steady regime of thermal conductivity with imposed boundary conditions which are: thermal flow and convection heat transfer. We observe that in table number one the maxim temperature equal 201.78 °C it is nod forthy and the minim temperature equal 46.96 °C it is placed in node one.

With the help of program TCIRTMEF we calculated the temperatures from the analysis domain at every time step, with an imposed initial condition. From fig.5 one notices that the temperature increases linearly with time.

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