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Generating Curves of Higher Order Using the Generalisation of Hügelschäffer's Egg Curve Construction

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Abstract: The starting constructions is the well known ellipse construction using the concentric circles c_1 and c_2. By eccentricity of the center C_2 for some value w, and for the preserved center of the transformation in the center C_1, the degree of the obtained curve rises from two to three, as done by mathematician Fritz Hügelschäffer. If we also displace the center of the transformation from C_1, we obtain a variety of higher order curves using the same principle in the constructive procedure.

Keywords: hyperbolism, circle, curve, ellipse, conoid.

1. INTRODUCTION

Graphic examination of the possibility of generating curves of higher order using the extension of Hügelschäffer's egg-shaped curve construction, bases upon constructive procedure of the Newton's transformation, known as hyperbolism [3], [9], [11]. The known construction of ellipse, Fig. 1a) is also based on the hyperbolism of two circles: c_1 (C_1, r_1) and c_2 (C_2, r_2). By distortion of this construction, Fig. 1b**)**, mathematician Fritz Hügelschäffer has performed the construction of egg-shaped curve (in 1946.), as he shifted the center C_1 off the center C_2 for the value w.

In the papers [7], [1], [8] and [6], we have already dealt with the spatial interpretation of the construction in question. Thus, in the paper [7] it has been demonstrated that, as a spatial model for this construction, we can use ruled surface - conoid. This surface has:

- directrix (d_1) - the intersection curve of rotating cylinder (with basis represented by c_1) and oblique cone (with basis represented by c_2, and generatices represent by radial rays of the transformation),
- directrix (d_2) - the straight-line set through the cone apex (V), parallel to y-axis (one of the directions of rays of transformation),
- directrix ($d_{3\infty}$) - the infinitely distant line of x-z plane, parallel to the generatrices of the conoid, so that their projections on x-y plane are seen as transformation rays of the x-direction. This is the case of the fourth order Conoid, which base is the combined curve of third order, cubic hyperbolic parabola with an infinitely distant line of x-y plane [1].

In [9] we have shown that the complement of complete cubic hyperbolic parabola, only a part of which is given by Hügelschäffer's construction, could be obtained by using two rectangular hyperbolas, complementary to the initial circles c_1 and c_1 [5]. In [6] we have noticed that the relocation of the center V (the cone apicis projection) off the center C_1, results with double egg curve, spatially interpreted as bases of two conoids: each formed from a portion of the degenerated space curve obtained by rotating cylinder and oblique cone intersection.

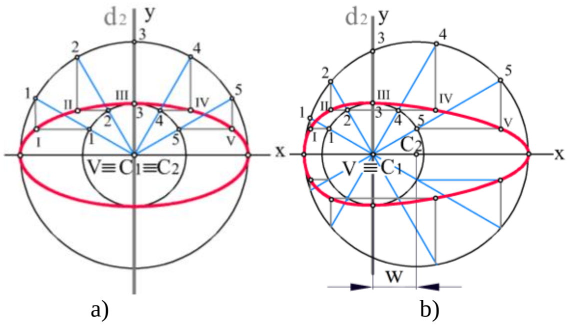

Fig. 1 a) Ellipse construction, b) Hügelschäffer's Egg curve construction

2. RELOCATON OF *V* IN *X*-DIRECTION

In this paper, we continue the extension of the Hügelschäffer's construction and the examination of possible shapes of such obtained curves.

Observe changing the shape of the curve (Fig. 2) by displacing the center C_1 off the center C_2, coincident in the initial construction of the ellipse, for the value w (Fig. 1b**)**. Such modified Hügelschäffer's construction will give the cubic egg curve, which form will vary depending on the ratio r_1/r_2 of the circles' radii, while retaining their type and order, except for the case where the center C_1 belongs to the circle c_2 (Fig. 14); then the cubic curve would degenerate to parabola and the straight-line.

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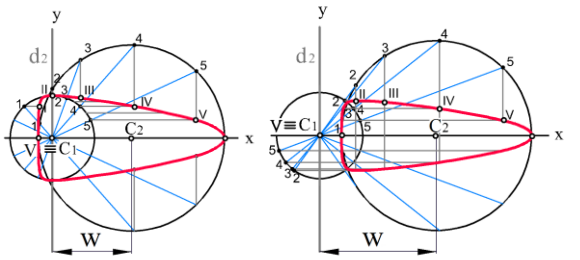
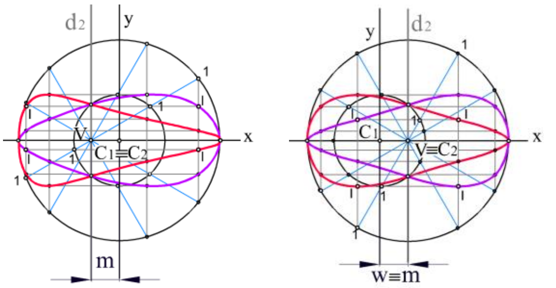


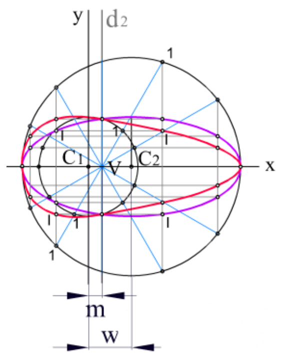
Fig. 2 Cubic egg curves changing shapes by moving the center $C_1 \equiv V$ along the x-axis

If we now displace the transformation center V off the center C_1 along the x-axis, for a value of m, we get double egg curve, even if the circles c_1 and c_2 are concentric (Fig. 3a). For the value of $w=m$, the two curves will be symmetrical (Fig. 3b), while for the case, as in Fig. 4, when the projection of the double straight line d_2 intersects the circle c_1 at the points corresponding to the intersection point of the ellipse centered at C_2 with the parameters $a=r_2$, $b=r_1$, this double cubic curve degenerates into ellipse and Granville's egg [6]. Thus, it has been shown that Granville's egg can be also obtained using the hyperbolism of two circles, not just by the hyperbolism of circle and line [11].



a) b)

Fig. 3 Double oval curves: a) for $w=0$, $m \neq 0$ b) $w=m$ (Granville's egg)



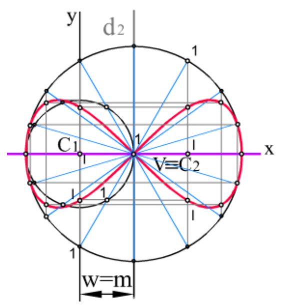


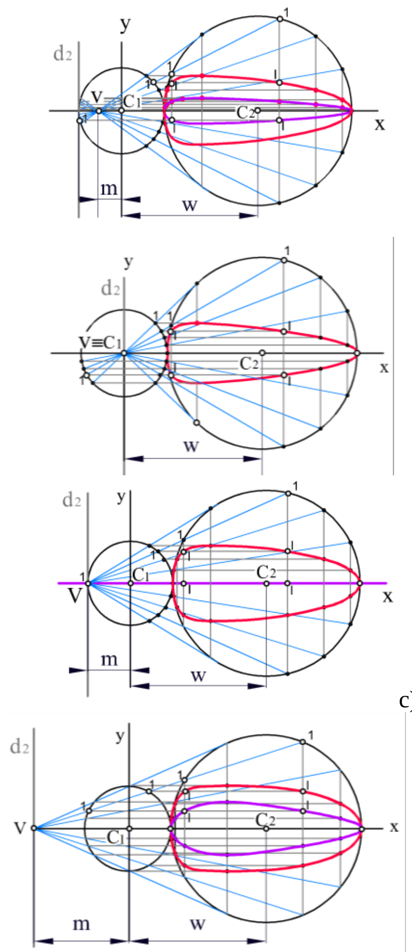
Fig. 4 (left) Double oval curves degenerating to Ellipse and Granville's Egg [6]

Fig. 5 (right) Lemniscate of Gerono

If we set the center V (the projection of oblique cone's apex) exactly on the circle c_1, we obtain the curve of the fourth order, because there will occur degeneration of the double ovals, due to the existence of multiple points in V. In case of center $V \equiv C_2$ belonging to the circle c_1 Fig. 5 ($2r_1=r_2$), we obtain lemniscate of Gerono and double straight-line x [12].

To visually track the rise of the curve's order by moving the center V off the center C_1, consider how translation of V along the x-axis influences the shape and the order of curve, Fig. 6a-d).

In the example given in Fig. 6a**)** $C_1 \equiv V$, it is a form Hügelschäffer's construction, with the circle c_1 now located outside the c_2 while touching it. The result will be, again, a cubic egg-shaped curve. If we move center V along the x-axis, for the value of m, we get a curve of order higher than three [6]. Because the hyperbolism, as a quadratic transformation, for the initial curves of orders ***n*** and ***p*** gives the new curve of order ***2np***, in the case of two curves, the curve order is 8^{th}. Hence, each curve obtained by this transformation can be treated as an octic, degenerated to curves of lower order (including the straight lines) which can also be overlapped. Exactly that will happen in case of Hügelschäffer's egg curve.



a) b) c) d)

Fig. 6 Three examples of different oval curves generating by moving the vertex V along the x-axis

Hence, non-degenerated curve of the eighth order cannot be obtained if we adopted two circles for the initial curves of the hyperbolism. As the spatial interpretation explains [1], there will always exist infinitely distant straight line of the plane x-y, participating in the totality of the order. It can be double, and in some cases even multiple, whereat in the finiteness there appear the curves of order eight diminished for the multiplicity of the infinitely distant line(s). This happens because the circles, both c_1 and c_2, are passing the absolute points on the infinitely distant straight-line q_∞ of the plane *x-y*, i.e. their real representatives are passing the pair of infinite points on the complementary hyperbolas' asymptotes [2]. These points, as common for both of the initial circles, will also be the points of the newly obtained curve by hyperbolism. With the infinitely distant points X_∞ and Y_∞, which represent the intersections of the lines d_2 and $d_{3\infty}$ and the line q_∞, those points will always make one real generatrix of the conoid, the additional line of the complete curve obtained by hyperbolism. Hence, in the finiteness, we will get the curve of maximum 7. order, while in special cases, when centers C_1 or C_2 are coincident with the center *V*, in the infinity there will appear the double straight line. In the case $C_1 \equiv C_2 \equiv V$, as for the ellipse, it will be fourfold line, whereat the ellipse itself is double, so the 8^{th} order of the curve is completed.

Also, in the Hügelschäffer's construction, as concerned graphically, there will come to the overlapping of the curve's points, due to the symmetry of the circle c_1 in relation to the center *V*, so we obtain the curve of 3^{rd} order and the infinitely distant line q_∞, because $C_1 \equiv V$. The rays from *V* will be just the tangents of the complementary hyperbola, intersecting it in the pair of infinite and infinitesimally close points. By the displacement of *V* for the value *m*, the condition of tangency is violated; the rays from the center *V* are intersecting the complementary hyperbolas just in single infinite points, so the one-fold line q_∞ appears.

As shown in Fig. 6b, by displacement of the center *V* for the value *m*, instead of a single egg-shaped curve, the two ovals will appear as follows: a curve of third order, within which is a curve of fourth order, which makes degenerated heptic curve with the line q_∞ supplementing it to the octic. In the case shown in Fig. 6c, when $m=r_1$, $V \in c_1$, the quartic curve compresses inward the cubic curve, overlapping the line *x*. In the case shown in Fig. 6d, where *V* is outside the circles c_1 and c_2, the line on the *x*-axis, from the previous example, again turns into egg-shaped quartic within already existing cubic curve.

The same transformation may be observed in the examples in Fig. 3a**).** (The case of circles c_1 and c_2 intersecting in the conjugate imaginary points, when the obtained curves intersect in the common points of the d_2 projection and c_1). The two ovals obtained are cubic hyperbolic parabola and Granville's egg.

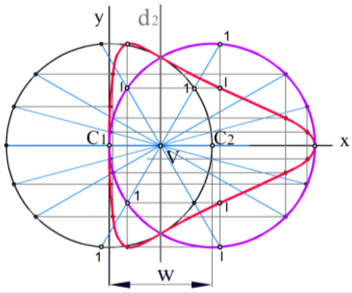


Fig. 7 Degeneration to quartic curve and the Circle

Special positions of some of the centers, or special ratio of the parameters *w*, *m*, r_1 and r_2, may cause degeneration of the curve; e.g. $V \in c_1$ or $V \in c_2$, causes degeneration of the curve with the appearance of the finite double (and sometimes quadruple) straight line.

Fig. 7 shows an example of the curve degeneration to a circle c_2 and the fourth order curve, where the circles c_1 and c_2 are identical, and the center *V* is at a distance $m=r_1/2=r_2/2$, which is the special case of the example shown in Fig. 4.

3. RELOCATON OF *V* IN *Y*-DIRECTION

The general case of hyperbolism of two circles, for the center *V(x,y)* is given in Fig. 8. Now we have translated the center of transformation *V*, not only in the direction *x*, for a value of *m*, but also in the direction *y*, for a value of *n*. Thus, we get the general shape of the curve of order *2(2x2)=8*, only part of which we obtain as the result of the construction.

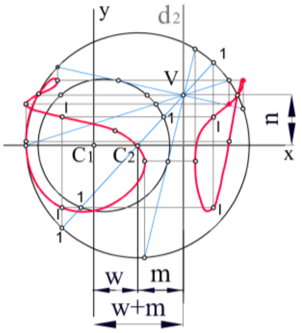


Fig. 8 General higher order curve obtained by extension of Hügelschäffer's construction

Such a curve is no longer symmetric with respect to the *x*-axis, yet the parallel asymptotes of the complementary hyperbolas still intersect at the common infinitely distant points, so the infinitely distant line q_∞ remains the part of the curve.

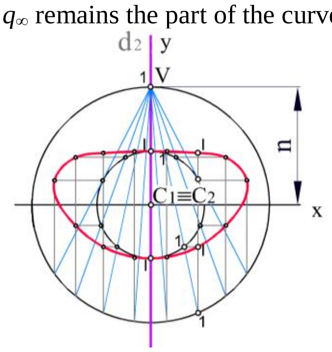


Fig. 9 Quartic egg curve

Fig. 9 shows the case of *w=0*, *m=0*, $n=r_2$, i.e. center $V \in c_2$. It comes to the degeneration of the curve to a quadratic egg curve, a double straight line - the projection of d_2 - and the infinite double line q_∞.

4. A CIRCLE OF INFINITE RADIUS - BACK TO THE CIRCLE AND LINE HIPERBOLIZAM

In respect with the above criteria, let us look at what happens in the special case, when one radius of the circles (r_1) participating in hyperbolism becomes infinitely large. Center (C_1) is now located at infinity, so we shall move the origin in the center C_2. This brings us back to the circle – line hyperbolism, the Newton's transformation. Maximum curve order in this case is $2(2x1)=4$.

In Fig. 10a, $w=\infty$, $m=0$, $n=0$. The resulting curve is Külp’s Quartic. Now there exists only one hyperbola, complementary to the circle c_2 of the finite radius, so in the infinity there will be neither common points of the curve, nor the curve degeneration that would result with the infinite line.

Fig. 10b) shows the previous example modified so that the position of the center V is set at the very circle c_2. Now there occurs the curve degeneration to the curve of the third order (the Witch of Agnesi) and the single line merged with the x-axis.

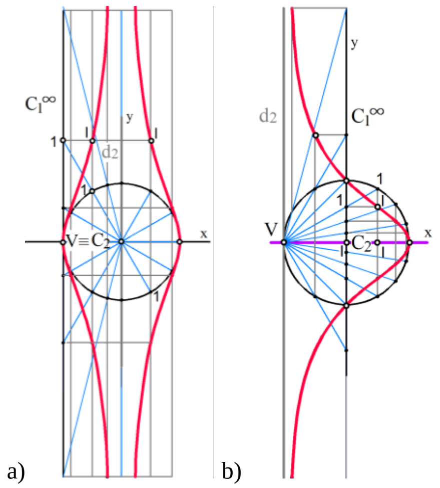


Fig. 10 a) Külp’s Quartic b) Witch of Agnesi

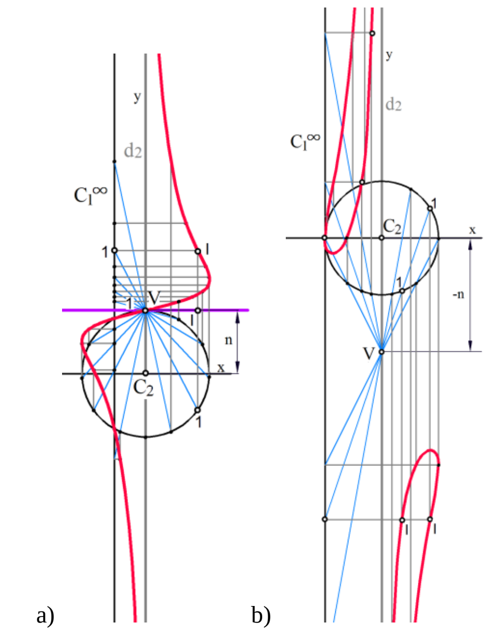


Fig. 11 a) Serpentine cubic, b) Two Branched Quartic

In the example shown in Fig. 11a**)** we have presented the generation of the serpentine cubic curve, with $w=\infty$, $m=0$, $n=r_2$; it comes to the curve degeneration to the cubic curve and the line, while in Fig. 11b**)** for condition $n\neq 0$, we obtain two branched quartic curve.

5. CONSTRUCTIONS OF CONICS BY MODIFICATION OF THE HÜGELSCHÄFFER’S CONSTRUCTION

In some special cases, instead of the curves of higher order, hyperbolism will result with conics. The conditions that are to be met derive from the above conditions of curve degeneration. As seen in Fig. 12a-**d)**, ellipse can be obtained even if some of the parameters w, m or n are different from zero, as in the case of switched rays’ directions (x, y) of the hyperbolism.

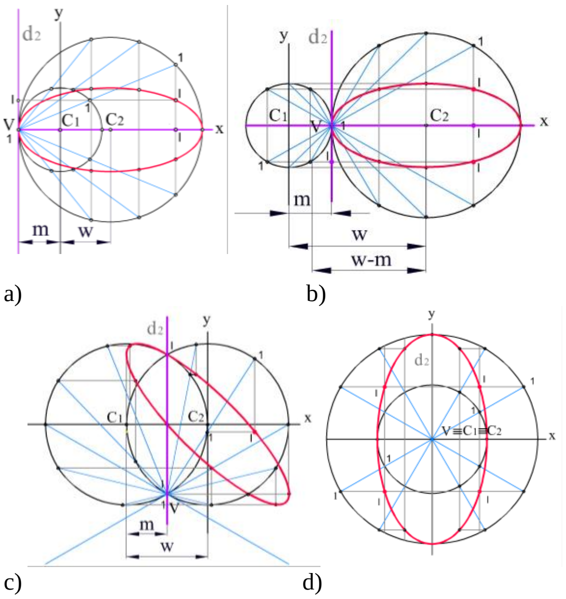


Fig. 12 Four Ellipse constructions based on hyperbolism of two circles

Thus, in Fig. 12a**)** we see that the ellipse occurs also when $w\neq 0$, $m=r_1$, $r_2=m+w$, $n=0$, where we get two more double straight lines, x and d_2 beside (now single) ellipse. In Fig. 12b**)**, $w=r1+r2$, $m=r1$, $n=0$, whereat the double straight lines x and d_2 also appear, along with the resulting ellipse. Ellipse can be get even when $n\neq 0$, $r_1=r_2$, $w=r_1/2=r_2/2$, and $n=r_1\sqrt{3}/2$, as shown in Fig. 12c**)**. Again, there appears additional (in this case quadruple) straight-line d_2, which means that in these cases, when $V\neq C_1$, and $C_1\neq C_2$, we get degenerated curve of 6^{th} order in the finiteness.

In Fig. 12d**),** it is shown the well known ellipse construction, only with the switched directions of the rays (x, y) of hyperbolism, i.e. for the case of $r_1>r_2$, which means that this change will not affect the order of the curve itself.

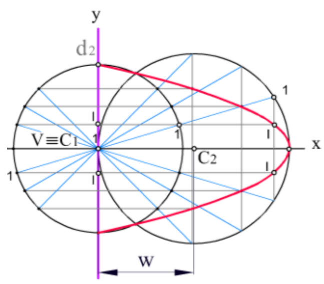


Fig. 13 Parabola

Parabola (Fig. 13) is obtained when the circle c_1 of the Hügelschäffer's construction is positioned so that its center C_1 is set on the circle c_2 [8]. Then $w=r_2$, $m=0$, $n=0$. We get a degeneration of cubic hyperbolic parabola, the curve that results the Hügelschäffer's construction, to the asymptotes of the curve: the parabola and the line merged with the *y*-axis.

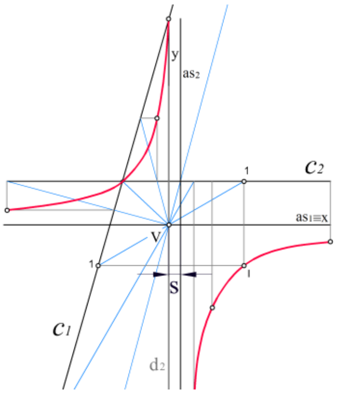


Fig. 14 - Hyperbola

Surely, we can get hyperbola using hyperbolism, Fig. 14, if we use two lines treated as the circles (c_1 and c_2) of infinitely large radii. Since the "curves" now are of the first order, i.e. the straight lines, the curve order will be $2(1x1)=2$.

6. THE SWITCH OF HYPERBOLISM'S RAYS

If we, as in Fig. 12d**)**, substitute the directions of rays (*x*, *y*) so that from the intersection points of the transformation rays with the circle c_1, we set lines of directions || *y*-axis, and from the points of intersection with the circle c_2, the lines of directions || *x*-axis, the curve order may change with respect to previous procedure, because now we measure the value of *m* from the center C_2.

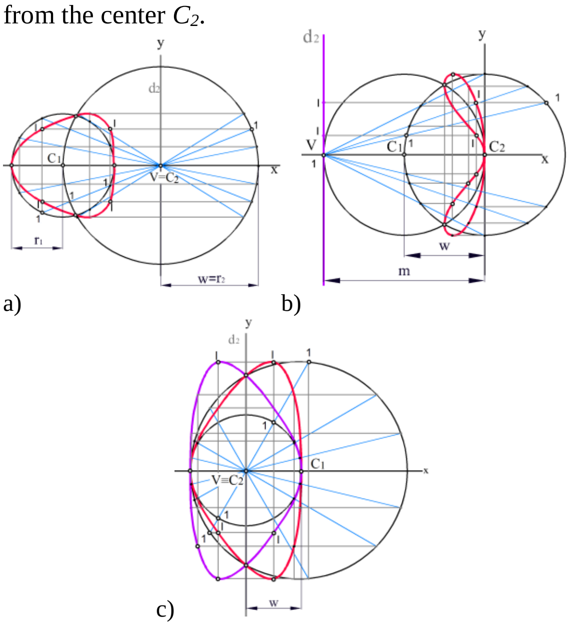


Fig. 15 Curves generated by the switched hyperbolism

Fig. 15a, b and **c)** show the curves of the third, fourth and sixth order, respectively.

As it can be seen, in the case of 15c**)** there will be no sefl-duplication of the curve, as in Hügelschäffer's construction, because now the circle c_2 takes the role of the one from which center the value *m* is measured, so it is also $m\neq 0$. In addition, spatially interpreted, the symmetry of the conoid's generatrices, which are now of *y*-direction, will no longer be in relation to the plane *x-z*, thus there is no overlapping of the curve in the front projection, where d_2 is seen as the point.

7. CONCLUSIONS

- Hügelschäffer's construction is based upon the well known Newton's transformation – hyperbolism.
- Based on this construction, by changing the parameter of distortion *w* and by displacement of the center *V* for a value *m* by x-axis and value *n* by *y*-axis, we can obtain different egg-shaped curves of higher orders, from cubic to octic.
- Each distance: *w, m, n* equal *0*, $r_1=m\pm w$, as well as $V\in c_1$, $V\in c_2$, implies degeneration of 8^{th} order curve.
- If one of the circles would have the infinite radius, and would be treated as line, we can obtain opened curves of 3^{rd} and 4^{th} order, with one or two parallel asymptotes.
- Conics can be constructed in several ways using hyperbolism, even if the parameters *w*, *m* or *n* differ *0.*
- Variety of curves' shapes generated by this method is best to be perceived by creation of a special applet, which will be the subject of further research.

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