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Generating a Type of Concave Cupolae of Fourth Sort

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Abstract: The paper discusses the generation of a specific group of polyhedra, Concave Cupolae of Fourth Sort (CC IV) with regular polygonal bases, using constructive and analytical procedures. Beside determination of the parameters of these polyhedra, the paper deals with their visualization, by the application of graphical software *MATLAB*. We consider one of the four possible types of forming the lateral surfaces of the Concave Cupolae of fourth sort.

Keywords: polyhedron, cupola, deltahedron, lateral surface, concave.

1. INTRODUCTION

Concave Cupolae are polyhedra which follow the method of generating the Johnson's cupolae (Johnson's solids [6]), where the convexity criterion is omitted, and in the lateral surface of these solids there appear two or more rows of equilateral triangles. The research of Concave Cupolae, as a polyhedral group, was initiated with the study of Concave Cupolae of Second Sort [1], [2], [3], [4], [5], and has been continued by the papers which dealt with Concave Cupolae of the higher sorts [7], [8], and [9]. The type of Cupola is dictated by the number of rows of equilateral triangles in the plane net of the polyhedron. Equilateral triangles are grouped into spatial hexahedral elements, whose polar arrangement around the central axis of the polyhedron makes a deltahedral lateral surface. The systematization of the properties of Concave Cupolae is presented in [9]:

1. Concave Cupola (**CC** hereinafter) is a regular faced polyhedron;
2. The lateral surface of **CC** is deltahedral;
3. Each edge of the **CC** is the joint of exactly two faces of the polyhedron;
4. The faces of **CC** can not penetrate or intersect each other, except by the edges;
5. The edges can not intersect elsewhere than in the vertices;
6. The planes of the **CC**'s faces can transect the interior space of the polyhedron – the lateral surface is concave;
7. **CC** is formed with the regular polygonal bases $\boldsymbol{\Omega_1}$ and $\boldsymbol{\Omega_2}$;
8. Base $\boldsymbol{\Omega_1}$ is the initial $\boldsymbol{n}$-gonal regular polygon, around which the deltahedral lateral surface is formed;
9. Base $\boldsymbol{\Omega_2}$ is $\boldsymbol{2n}$-gonal regular polyhedron;
10. Bases $\boldsymbol{\Omega_1}$ and $\boldsymbol{\Omega_2}$ lie in parallel planes;
11. The axis of **CC** is orthogonal to the planes of the bases $\boldsymbol{\Omega_1}$ and $\boldsymbol{\Omega_2}$;
12. Each side of the **CC** is visible from the exterior – there are no internal or hidden sides;
13. **CC** can not have adjacent coplanar sides.

2. CONCAVE CUPOLAE OF THE FOURTH SORT

The net of Concave Cupola of the fourth sort (**CC IV**, hereinafter) consists of the lateral surface (the fourfold strip of equilateral triangles), $\boldsymbol{n}$ separate supplemental spatial four-faced elements, and two base polygons: $\boldsymbol{\Omega_1}$ and $\boldsymbol{\Omega_2}$. The segment of the lateral surface net of **CC IV** is shown in the Fig. 1.

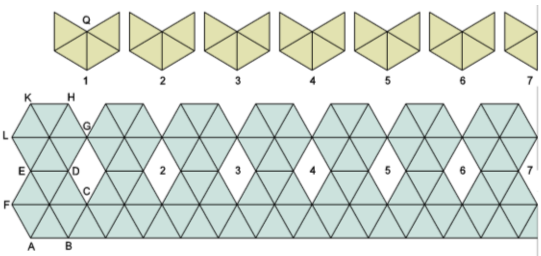

Fig. 1 – Fragment of the lateral surface of CC IV

In the net of the **CC IV** lateral surface, between the spatial hexahedral elements, in the blanks numbered from $\boldsymbol{1}$ to $\boldsymbol{n}$, it is necessary to insert additional spatial elements composed of four equilateral triangles, grouped around a common vertex $\boldsymbol{Q}$. Another way of forming the planar net of the lateral surface is such that it consists of complete four series of equilateral triangles (with no blanks), totalized by $\boldsymbol{n}$ pairs of equilateral triangles, subsequently added. In this paper, we chose the first method of forming the net of **CC IV**, in order to apprise the generating principle of the lateral surface, using the polar array of its unit cell.

Unlike **CC II** [1], the unit cell of **CC IV** consists of two (instead of one) spatial hexahedral elements, which are constituted of six equilateral triangles grouped around the common vertex (Fig. 2). The spatial hexahedral element $\boldsymbol{ABCDEFO_1}$ which participates in the composition of the lateral surface's lower belt, closer to the $\boldsymbol{\Omega_2}$, adjoins the hexahedral

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element ***EDGHKLO***$_2$, closer to the Ω_1. Equilateral triangles, which make spatial hexahedral element, are grouped around the common vertex O_1 in the lower belt, and O_2 in the upper belt of the **CC IV** lateral surface.

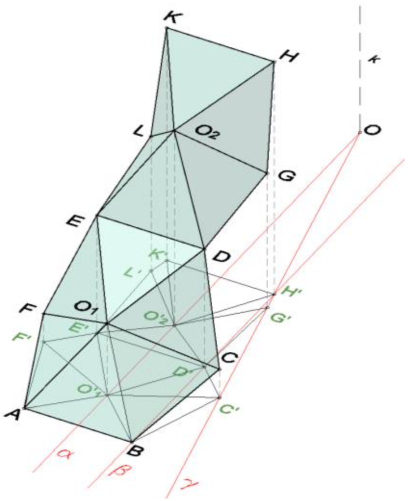

Fig. 2 – Unit Cell of CC IV-mm

There are four types of **CC IV** with the same polygonal base [9] depending on the positions of the vertices O_1 and O_2 (as seen from the exterior):

- **CC IV-Mm,** indented vertex O_2, and protruded vertex O_1.
- **CC IV -mm,** both vertices O_2 and O_1 protruded.
- **CC IV -mM,** protruded vertex O_2, and indented vertex O_1.
- **CC IV -MM,** both vertices O_2 and O_1 indented.

Generation of **CC IV-mm** is the subject of this paper (the generation of **CC IV-Mm** is presented in [8]).

3. GENERATION OF **CC IV-mm**

The constructive-geometric determination of the parameters of **CC IV** is based upon the observation of movement of the lateral surface's unit cell around the fixed edge ***AB***, whereat trajectories of **CC's** vertices are obtained. The positions and the heights of the vertices are determined by the intersections of the vertical planes **α**, **β** and **γ** (which contain the vertices of the unit cell) and spheres of radii $r=a$, where a is the side of constituent equilateral triangle. The centers of the spheres are located in the adjacent vertices of **CC IV**.

1. $\gamma\cap M_1(O_1;a)=c_1$, $\gamma\cap M(B;a)=c$, $c_1\cap c=C$
2. $\beta\cap M_2(C;a)=c_3$, $\beta\cap M_1(O_1;a)=c_2$, $c_3\cap c_2=(D,B)$
3. $\gamma\cap M_3(D;a)=c_4$, $\gamma\cap M_2(C;a)=c_5$, $c_4\cap c_5=Q$
4. $\gamma\cap M_5(Q;a)=c_6$, $c_4\cap c_6=(G, C)$
5. $\alpha\cap M_6(G;a)=c_9$, $\alpha\cap M_3(D;a)=c_8$, $c_8\cap c_9=O_2$
6. $\beta\cap M_7(O_2;a)=c_{10}$, $\beta\cap M_6(G;a)=c_{11}$, $c_{10}\cap c_{11}=(H,D)$

as is shown in **Fig. 3**

Let us iterate the constructive procedure (shown in Fig. 3) in order to get the approximate trajectory of the vertex ***H***, depending on the position of the preliminary adopted vertex O_1. We always choose the vertex O_1 to belong to the plane **α** and be above the horizontal plane of the base Ω_2. By the intersection of the trajectory of the vertex ***H*** and the vertical plane (***υ***), on which we expect to find the position of the vertex ***H***, we obtain the final position of the vertex ***H***. Using retrograde constructive steps (Fig. 4) we get the remaining vertices of **CC IV-mm** unit cell, i.e. of the spatial hexahedrals ***ABCDEFO***$_1$ and ***EDGHKLO***$_2$.

1. $\gamma\cap M_7(O_2;a)=c_{13}$, $\gamma\cap M_8(H;a)=c_{14}$, $c_{13}\cap c_{14}=G$
2. $\beta\cap M_9(G;a)=c_{15}$, $\beta\cap M_7(O_2;a)=c_{16}$, $c_{15}\cap c_{16}=D$ i H
3. $\gamma\cap M_{10}(D;a)=c_{17}$, $\gamma\cap M_9(G;a)=c_{18}$, $c_{17}\cap c_{18}=Q$
4. $\gamma\cap M_{11}(B;a)=c_{19}$, $c_{19}\cap c_{17}=C$
5. $\alpha\cap M_{10}(D;a)=c_{20}$, $\alpha\cap M_{11}(B;a)=c_{21}$, $c_{20}\cap c_{21}=O_1$

In the given procedure for the graphical determination of vertices of the spatial hexahedral elements ***ABCDEFO***$_1$ and ***EDGHKLO***$_2$ we obtain two solutions for each of the vertices ***C***, ***Q*** and O_2: one with the major (**M**), and one with the minor (**m**) height. By the variant of the constructive procedure which adopts lower heights for the vertices ***C*** and ***Q***, and the greater height for the vertex O_2 (Fig. 3 and Fig. 4) **CC IV–mm** is generated.

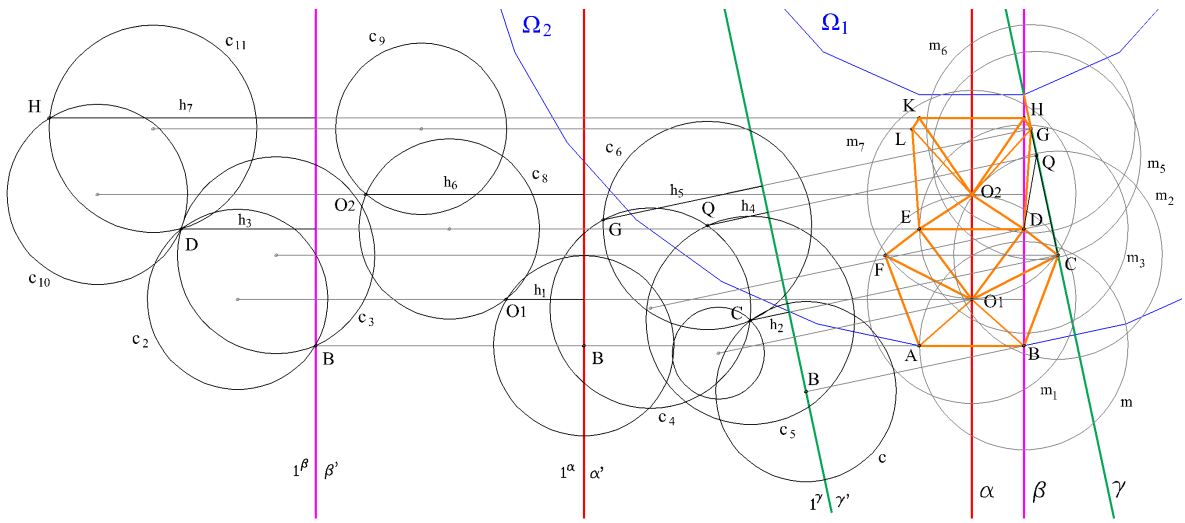

Fig. 3 – Finding the positions and the heights of vertices of CC IV-mm unit cell, for the adopted position of the vertex O_1

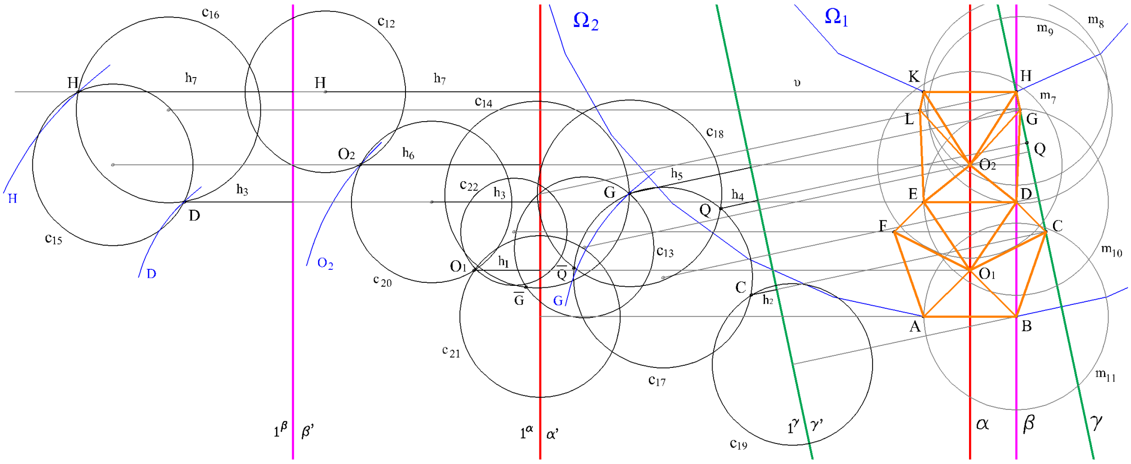

Fig. 4 – Determining the trajectories of the vertices of CC IV–mm unit cell

In the described constructive procedure, we have found the positions and the heights of the vertices of **CC IV-15mm,** by the intersection of the planes **α, β** and **γ** with curves of the higher order – the trajectories of the vertices, obtained by changing the position of the vertex $\boldsymbol{O_1}$ of the hexahedral cell ***ABCDEFO₁***.

The presented procedure was the basis for the creation of the algorithm [9] solving of which (by the application of iterations in *Microsoft Excel*) it has been enabled to calculate all the metric relations and parameters for the direct reading of the measurements of **CC IV -mm**.

Input data:

n **–** number of the vertices in the base polygon $\boldsymbol{\Omega_1}$

a **–** side length of the base polygon $\boldsymbol{\Omega_1}$

Δ– expected error after the iteration procedure is performed.

The height $\boldsymbol{h_1}$ (of the vertex $\boldsymbol{O_1}$) is levelled by the iterative procedure (and consequently all the other parameters) till the expected error is achieved, after the iteration procedure is performed. More about the algorithm, see in [9].

Based on the iteratively obtained parameters in the software package *MATLAB*, a program for modeling the lateral surface of **IV CC-mm** has been created. For each vertex, there were determined cylindrical coordinates in relation to the origin, adopted in the center of the basic polygon $\boldsymbol{\Omega_2}$. The values of the radii, angles and heights of the adopted initial positions of the vertices of **CC IV-mm** unit cell, i.e. hexahedral elements ***ABCDEFO₁*** and ***EDGHKLO₂***,have been obtained by the iterative procedure. The lateral surface of **CC IV-mm** is generated by polar array of the unit cells around the central axis of the cupola, whereat every point with the same denotation (e.g. $\boldsymbol{D_1}$ and ***D***) is distant from the previous one for the angle: $\beta=2\boldsymbol{\pi}/\boldsymbol{n}$, which is the amount of the growth angle in cylindrical coordinates for each vertex.

Using the created program, for modeling the lateral surface of **CC IV-mm** in the *MATLAB* software package, we investigated the smallest possible number of the sides of the base $\boldsymbol{\Omega_1}$ that enables the generation of **CC IV**, so that the formed polyhedron retains all the features of Concave Cupola defined in the introductory section of this paper. For base polygons $\boldsymbol{\Omega_1}$ whose number of sides $\boldsymbol{n \leq 7}$, the faces of the lateral surface penetrate each other – so the conclusion is that the octagon is the smallest polygon which enables the generation of **CC IV-mm**.

In further analysis, we defined two subgroups of **CC IV-mm,** depending on the number of the base $\boldsymbol{\Omega_1}$ sides. The first subgroup includes **CC IV-mm** with basic polygons which number of sides is in the range $\boldsymbol{8 \leq n \leq 13}$**.** There is a clear differentiation between the lower and the upper belt of the lateral surface. The lateral surface of **CC IV-11mm**, simulated by *MATLAB* software package is shown in Fig. 5.

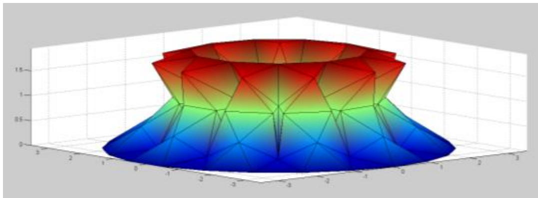

Fig. 5 – Lateral Surface of CC IV -11mm

The second **CC IV-mm** subgroup consists of the Cupolae with polygons $\boldsymbol{n=14}$, $\boldsymbol{n=15}$ and $\boldsymbol{n=16}$**.** The form of the cupola's lateral surface is substantially changed; there is no clear differentiation between the lower and upper belts, while **CC IV** itself takes the shape of the cupola (dome) in the architectural sense.

CC IV-15mm lateral surface is shown in Fig. 6.

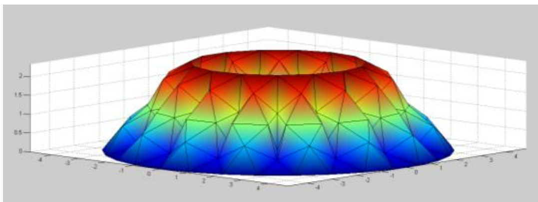

Fig. 6 – Lateral Surface of CC IV -15mm

For polygons which number of sides is in the range of $\boldsymbol{17 \leq n \leq 21}$ it is possible to generate lateral surface, but not **CC IV** itself, because the lateral faces penetrate the basis $\boldsymbol{\Omega_2}$. I other words, the common vertex ***Q*** of the constituent four-faced deltahedral element of the upper belt, passes below the base plane $\boldsymbol{\Omega_2}$ of the cupola. If we adopt the position of the vertex ***Q*** in the ground horizontal plane, the lateral deltahedral surface generated in such manner can find its application in engineering practice, as shown in Fig. 7**,** for **CC IV -17mm**.

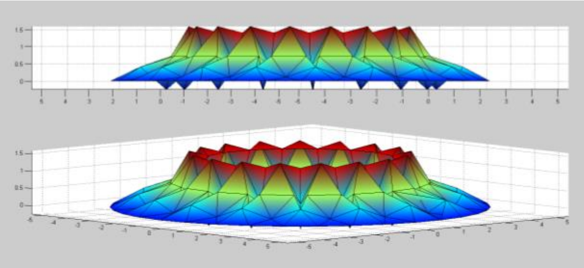

Fig. 7 – Front view (above) and 3D view (below) of the deltahedral lateral surface of CC IV-17mm.

Tab. 1 **–** Cylindrical coordinates of the unit cell's vertices of **CC IV-mm** for $\boldsymbol{n}$=11, $\boldsymbol{n}$=15 and $\boldsymbol{n}$=17 ($\boldsymbol{a}$=1, $\Delta=1^{-10}$)

vertex	n	radius	angle	height
A	11	3.5133	8.1818	0.0000
	15	4.7834	6.0000	0.0000
	17	5.4190	5.2941	0.0000
B	11	3.5133	24.5455	0.0000
	15	4.7834	18.0000	0.0000
	17	5.4190	15.8824	0.0000
C	11	2.6228	32.7273	0.1390
	15	3.9388	24.0000	0.2833
	17	4.6331	21.1765	0.4102
D	11	2.1211	29.9984	0.9967
	15	3.5558	20.0834	1.1709
	17	3.7054	18.3433	0.7222
E	11	2.1211	2.7289	0.9967
	15	3.5558	3.9166	1.1709
	17	3.7054	2.8332	0.7222
F	11	2.6228	0.0000	0.1390
	15	3.9388	0.0000	0.2833
	17	4.6331	0.0000	0.4102
$\mathbf{O_1}$	11	2.7623	16.3636	0.4883
	15	4.2473	12.0000	0.7000
	17	4.5374	10.5882	0.1142
G	11	1.1238	32.7273	1.0008
	15	2.5931	24.0000	1.3446
	17	3.1219	21.1765	0.4102
H	11	1.7747	32.7273	1.9446
	15	2.4049	24.0000	2.3267
	17	2.7211	21.1765	1.4023
K	11	1.7747	0.0000	1.9446
	15	2.4049	0.0000	2.3267
	17	2.7211	0.0000	1.4023
L	11	1.1238	0.0000	1.0008
	15	2.5931	0.0000	1.3446
	17	3.1219	0.0000	0.4102
$\mathbf{O_2}$	11	2.5486	16.3636	1.7584
	15	3.1288	12.0000	1.9433
	17	3.1774	10.5882	1.5882
Q	11	1.6228	32.7273	0.1342
	15	2.9468	24.0000	0.4092
	17	3.8775	21.1765	-0.2449

Cylindrical coordinates of the initial unit cell, for generating the lateral surface (of cupolae in Fig. 5-7) in the *MATLAB* software package, are presented in Tab. 1. The adopted edge length $\boldsymbol{a}=1$ and the expected error after the performed iteration procedure is $\boldsymbol{\Delta}=1^{-10}$.

The basic parameters for **CC IV-11mm**, **CC IV-15mm** and **CC IV-17mm** deltahedral lateral surface, based on the constructive procedure for generating **CC IV-17mm**, are shown in Tab. 2.

Tab. 2 – The basic parameters of **CC IV -mm**

	vertices	edges	faces
CC IV -11mm	110	297	189 (cupola)
CC IV -15mm	150	405	257 (cupola)
CC IV -17mm	170	459	289 (lateral s.)

4. CONCLUSIONS

The polyhedra shown in this paper represent a type of Concave Cupolae of Fourth Sort from the infinite family of Concave Cupolae. By variant of the constructive procedure with the adopted lower heights of the vertices **C** and **Q,** and greater height of the vertex $\boldsymbol{O_2}$, Concave Cupolae of Fourth Sort (**CC IV-mm)** are generated. The Unit cell of **CC IV-mm** is characterized by protruded vertices $\boldsymbol{O_1}$ and $\boldsymbol{O_2}$, common vertices of equilateral triangles within the spatial hexahedral elements $\boldsymbol{ABCDEFO_1}$ and $\boldsymbol{EDGHKLO_2}$. **CC IV-mm** can be developed with a basic polygon $\boldsymbol{\Omega_1}$ which number of sides spreads in the range of 8≤$\boldsymbol{n}$≤16. For polygons with 17≤$\boldsymbol{n}$≤21 it is possible to form the lateral surface, but not the cupola itself. The analyzed polyhedral structures, or just parts of them, may find place in architectural practice, which will be the subject of further research.

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