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Geometric principles for the study of repeated patterns in one and two dimensions

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Abstract: This paper identifies geometric concepts and constructions which are of great value to design practitioners. Particular attention is focused on geometric symmetry as the basis of an analytical tool to examine designs in different cultural contexts. In one dimension, ornaments of stripes are classified by seven groups, which are systematically produced by periodic translations in one direction and reflection transverse to the longitudinal axis of translations. In two dimensions, there are seventeen wallpaper groups, produced by translations in two directions, reflections, inversions and rotations. Plane patterns can be classified according to the symmetries they admit. In this paper the symmetries of the plane, and the resulting classification of patterns, are given and described.

Keywords: symmetry, plane group, plane pattern, translation, reflection, rotation.

1. INTRODUCTION

From the ancient times, until recently, geometry was the design tool for engineers, builders, artists and designers. An understanding of the basic principles of Euclidean geometry can offer immense potential in addressing and solving design problems in recent times. Various geometric characteristics, concepts, constructions, comparative measures and ratios are of particular importance to both the design practitioner and the design analyst [4].

The term „symmetry” is the Greek for proportionality and similarity in arrangement of parts, symmetry being a kind of relation between parts like equality of parts, harmony or equivalence of parts, regularity in the arrangement of parts.

An object or structure is symmetrical if it looks the same after a specific of change is applied to it [7]. There are a wide variety of applications of the principle of symmetry in art, in the inorganic and organic nature. It is seen in various forms: bilateral symmetry, translation symmetry, rotational symmetry, ornamental symmetry, crystallography symmetry etc.

This paper focuses on geometric concepts for the study of repeated patterns. The intention is to explain the nature of geometric symmetry, to highlight its potential value as an analytical tool to the design researcher and to show how an understanding of the relevant principles can also aid the design practitioner.

2. GEOMETRIC CONCEPTS

2.1 The distance-preserving transformations of the plane onto itself

We live and work in a symmetrical world. In fact the vast majority of living creatures, manufactured objects, tools and utensils, constructions, monuments, exhibit various forms of symmetry. In the bilateral symmetry, two components and equal parts are each a reflection of the other [1].

An object is symmetric with respect to a given mathematical operation, if, when applied to the object, this operation does not change the object. Two objects are symmetric to each other with respect to a given group of operations if one is obtained from the other by some of the operations (and vice versa) [2].

There is a considerable amount of literature dealing with the mathematics behind wallpaper patterns. The basic tool for the study and creation of repeated patterns is symmetry. Synonyms are rigid motion or isometry, which are often used to emphasize the defining property of symmetry, which is that symmetry is a distance-preserving transformation of the plane onto itself [3].

An isometry of an object or space is any movement of the object or space which doesn’t change the distances between the points of that object or space.

"Rigid motion" involves only the initial position and the final position of the points of the plane.

Familiar examples are rotation about a given point by a given angle, translation in a given direction by a given distance, and reflection in a given line [10].

A translation is an isometry which is a shift of some specified direction and distance. The vector which describes a translation has direction vertically, horizontally (fig. 1) or diagonally.

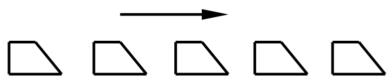

Fig. 1 Translation symmetry

There is an exact correspondence between vectors and translations: every vector in a plane defines a

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unique translation, and every translation of a plane onto itself can be described by some vector in that plane.

A rotation is another isometry, determined by a center and an angle (fig. 2). Rotation allows a motif to submit repetition at regular intervals around an imaginary fixed point, known as a center of rotation.

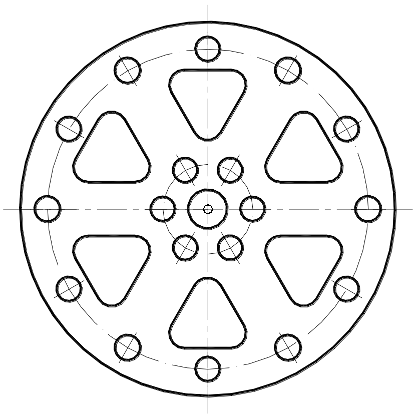


Fig. 2 Spoked membrane with rotational symmetry

The word motif means a repeated design element. Mathematicians often use this word to refer to the smallest portion of a pattern that can be repeated to recreate the entire pattern. Motifs are the building blocks from which patterns are produced.

A symmetrical motif is comprised of two or more parts, of identical size, shape and content.

A reflection is an isometry specified by a line of reflection. If a straight line is drawn on a plane and the plane is folded along this line then every point *A* on one side of the line lies against some point *A*' on the other side of the line [4].

The transformation which takes each point *A* to its corresponding point *A*' is a reflection in the line.

A description of the relation between point *A* and their image *A*' is that if a perpendicular segment is dropped from *A* to the line on the (unfolded) plane and then extended the same distance on the other side of the line, the end of the extended segment is *A*'.

Fig. 3 shows the reflection of four points *A*, *B*, *C*, *D* in a line *l* to their four images *A*', *B*', *C*', *D'*.

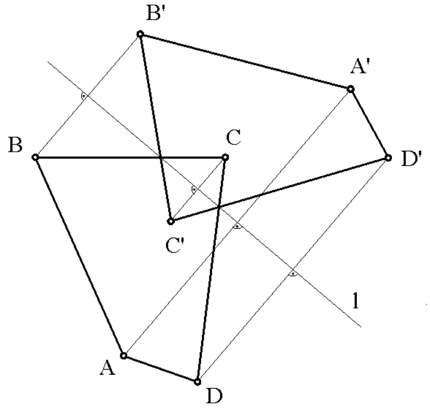


Fig. 3 Reflection in line *l*

The product, or composition, of two isometries is the isometry resulting from applying one and then the other in order [6].

A glide reflection is the product of a reflection and a translation along the line of reflection (fig. 4).

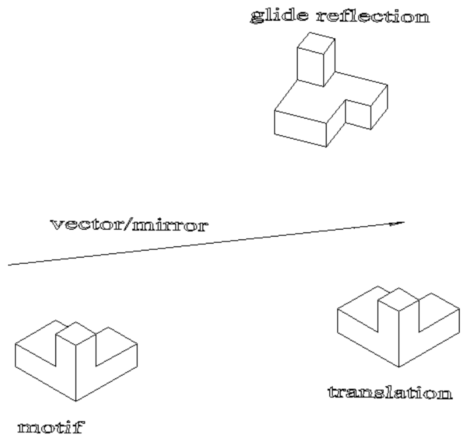


Fig. 4 Glide reflection-specified by a vector and a parallel mirror line

This produces a "footprint" pattern. If every point *P* in the plane is translated by a given vector *v* and then reflected in a line *l* parallel to *v* to point *P*' the transformation taking *P* to *P*' is called a glide reflection, and the line *l* is the axis of the glide reflection.

The footprints in fig. 5 suggest glide reflection. Each foot on one side of the line is translated by vector *v* and then reflected to the next foot on the other side of the line.

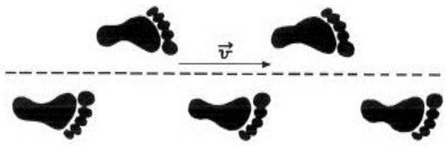


Fig. 5 Glide reflection [3]

Symmetry of a pattern is any isometry that leaves the appearance of the pattern unchanged. Symmetry is created with the movement of a motif by one of four methods (rotation, translation, reflection and glide reflection) that preserve the relationships of distance, size, angles and shape.

An infinite strip with a repeating pattern is called a frieze pattern, or sometimes a border pattern or an infinite strip pattern [11]. The term "frieze" is from architecture, where a frieze refers to a decorative carving or pattern that runs horizontally just below a ceiling.

Border patterns exhibit translation of a motif at regular intervals in one direction only (fig. 6).

Fig. 6 Frieze pattern [12]

Given that every isometry is composed of one, two or three reflections, the composition of two reflections is a translation (if the reflection lines are parallel) or a rotation (about their point of intersection, if the two lines intersect), while the composition of three reflections is either a reflection or a glide reflection [2].

For example, if a circle is drawn with a compass, then any rotation about the center of the circle takes every point of the circle onto another point of the circle.

If the drawing of footprints shown in fig. 5 extends forever to the left and right then a translation by vector 2*v* takes every point of one footprint to a point on another footprint.

In the first case we say that the drawing admits a rotation, or "rotation about the center of the circle is an isometry of the drawing".

In the second case we say that the drawing admits a translation, or specifically, "translation by 2*v* is an isometry of the drawing".

Depending on the constituent symmetry characteristics, motifs may be classified using the notation *cn* (*c* for cyclic) or *dn* (*d* for dihedral). Motifs from family *cn* have *n*-fold rotational symmetry and motifs from family *dn* have *n* distinct reflection axes as well as *n*-fold rotational symmetry. Relevant illustrations are provided in figure 7.

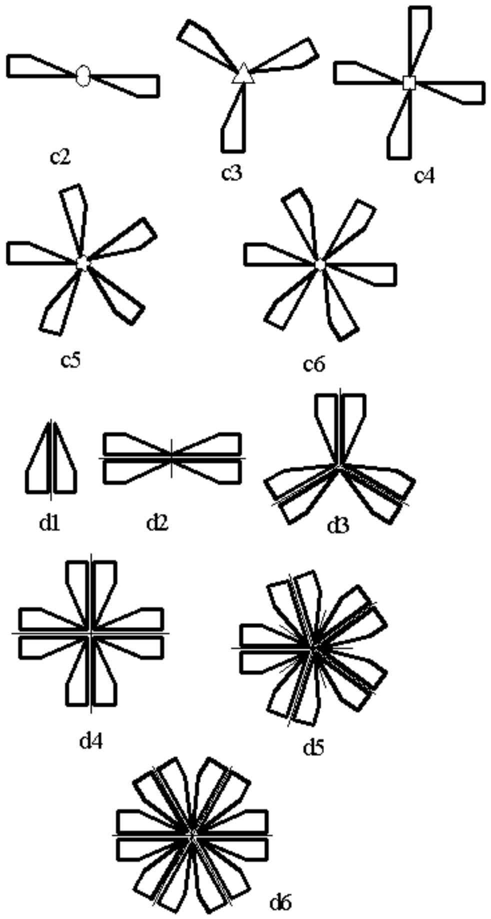


Fig. 7 Schematic illustrations of classes *cn* and *dn* motifs

The majority of divisions from family *cn* can be seen all around us. Those that are based on simple numbers, or their multiples, are easy to find, but patterns based on larger numbers are more difficult to construct and to find.

Fig. 8 shows a tile which is constructed with nineteen divisions, having *19*-fold rotational symmetry, also known as a nonadecagon. This pattern is symmetrical about its vertical axis but not its horizontal axis.

Fig. **8** Nineteen – fold rotational symmetry [5]

2.2 Determination of dimensionality

Symmetry of plane patterns is a special kind of geometrical symmetry [8].

Figures like the circle or ellipse which do not admit any translation are called finite.

Figures like the string of footprints, which admit translations in one direction (and its opposite) only are called one-dimensional, or more briefly, bands or strips. Many classical friezes, wallpaper designs, rows of windows in high-rise apartment buildings exhibit this type of symmetry.

Figures which admit translations in two directions are called two-dimensional. Two translation vectors are marked in fig. 9. Note that if there are translations in two directions then there are translations in infinitely many directions, since the two translation vectors can be combined in infinitely many ways.

Thus dimensionality is completely determined by the nature of the translations which a figure admits.

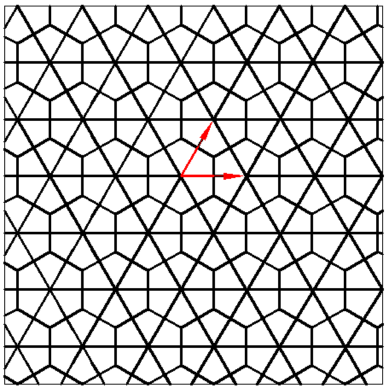


Fig. 9 Two-dimensional pattern showing translation vectors in two directions

3. THE SEVEN ONE-DIMENSIONAL PATTERNS

There are a limited number of kinds of isometry which are admitted by a one-dimensional pattern.

By definition, there are translations in only one direction (and it’s opposite). This means that if there is a glide reflection it must be in the same direction as the translations.

Moreover, any reflection must be in a line in the direction of the translation (which we call "horizontal") or perpendicular to that direction (i.e. "vertical") so there can be only these two kinds of reflection.

Finally, the only possible rotation is a half-turn (180°), which interchanges the translation vector with its opposite.

By comparison from the translations (by *v* and its multiples) which are present in every one-dimensional pattern there are only four possible kinds of isometry for a one-dimensional pattern: reflection in a line parallel to *v* ("horizontal reflection"), reflection in lines perpendicular to *v* ("vertical reflection"), half-turns and glide reflection.

Note that the term "horizontal reflection" is used as an abbreviation for "reflection in the central axis of the infinite band", and "vertical reflection" refers to "reflection in a line perpendicular to the direction of the band".

For one dimensional periodic arrangement, there are seven different symmetries, which are systematically produced by translations in one direction and reflections transverse to the longitudinal axis of translations (fig. 10):

1. translational symmetry only, in the horizontal direction – symbol *p1*;
2. longitudinal mirror plane (translation, horizontal line reflection and glide reflection – symbol *p11m*);
3. C_2 axis in combination with translations (translation and vertical line reflection –symbol *p2m1*);
4. perpendicular mirror plane (translation and 180° rotation – symbol *p2*);
5. glide reflection and perpendicular mirror plane translation, 180° rotation, vertical line reflection and glide reflection – symbol *p2mg*);
6. glide reflection (translation followed by a reflection in line parallel to the direction of translation – symbol *p11g*);
7. longitudinal and perpendicular mirror plane (translation, 180° rotation, horizontal line reflection, vertical line reflection and glide reflection – symbol *p2mm*).

Each of the seven patterns has a symbol name of the form *pxyz*, obtained as follows:

- the letter *p* prefaces each of the seven;
- the letter *x* is the symbol which denotes symmetry operations perpendicular to the longitudinal axis of the border; *m* is used where vertical reflection is present, or the number *1* where the operation is absent;
- the symbol *y* denotes symmetry operations working parallel to the sides of the border; the letter *m* is used if longitudinal reflection is present, the letter *a* if glide-reflection is present or the number *1* if neither is present;
- the fourth symbol *z* denotes the presence of two-fold rotation; the number *2* is used if rotation is present and the number *1* if rotation is not present.

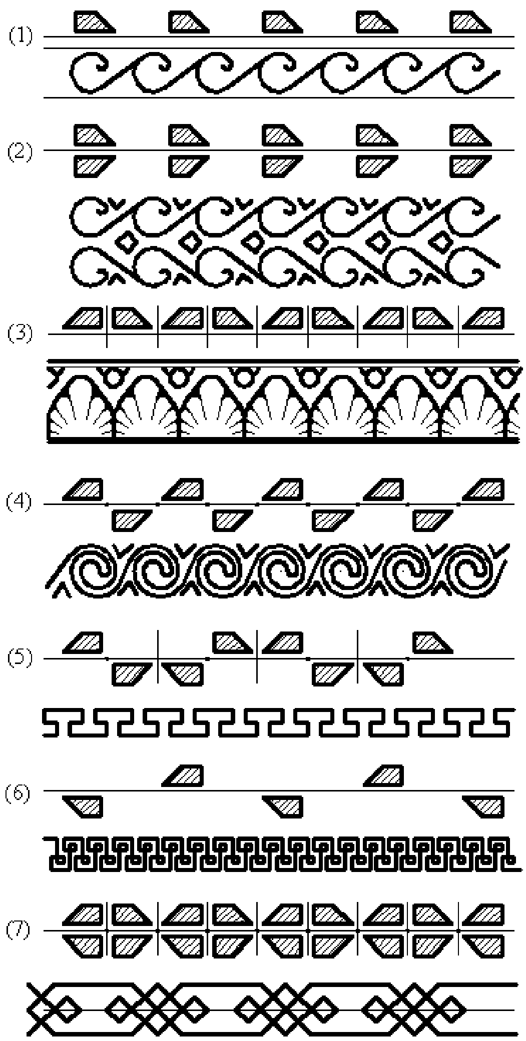


Fig. 10 Symmetries in one dimension
Examples of all seven one-dimensional patterns

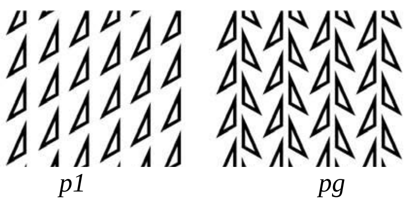
4. THE SEVENTEEN TWO-DIMENSIONAL PATTERNS

Plane patterns are characterized by translation in two independent directions across the plane.

In the inorganic world, crystals, there are possible only rotational symmetry of order 2, 3, 4 and 6. Snow crystals offer the best known examples of hexagonal symmetry. For any pattern which admits translations in more than one direction the only possible rotations are by 180° (order 2), 120° (order 3), 90° (order 4), and 60° (order 6), that is, by a half-turn, third of a turn, quarter-turn, or a sixth of a turn [4].

As a consequence of the restriction on rotations, the angles between lines of reflection (or glide-reflection) are also restricted.

The analysis of restrictions shows that in fact there are only 17 combinations, and hence only 17 two-dimensional patterns (fig. 11).



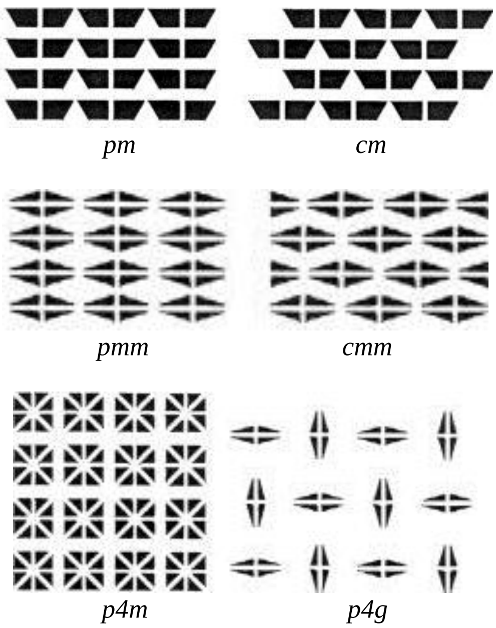


Fig. 11 Schematic representations of a group of eight out of seventeen two-dimensional patterns [3], [4]

The way in which symmetrical patterns are produced and categorised is determined by the mathematics of group theory. Associated with these seventeen classes is a notation, which identifies the highest order of rotation within the pattern together the presence (or absence) of glide-reflection and/or reflection. The notation used in this paper is the IUC notation adopted by the International Union of Crystallography in 1952, based on crystallography. Other system of notation is the orbifold notation, based not on crystallography, as is the IUC system, but on topology. Each of the 17 patterns created as covering on walls, floors and ceilings has a three-symbol name, obtained as follows:

- a number occurring in the name indicates the presence of the corresponding rotation;
- the occurrence of *m* or *g* indicates the presence of reflections (*m* for mirror, *g* for glide reflection);
- the letter *c* indicates a "half-drop".

There are five groups in which patterns can be arranged together to give different effects:

- the first group are made without rotation and are known as: *p1, pm, pg* and *cm*;
- the second group are constructed using rotations of 180°, without rotations of 60° or 90°: *p2, pgg, pmm, cmm* and *pmg*;
- the third group are constructed with rotations of 90°: *p4, p4g* and *p4m*;
- the fourth group use rotations of 120°: *p3, p3ml* and *p3lm*;
- the fifth group are constructed with rotations of 60°: *p6* and *p6m*.

The diagrams shown in fig. 12 - 13 describe a method for determining the category into which a particular symmetrical pattern might divide, that starts with the number of rotations to be found in the pattern. Determining a pattern type is based on the question whether or not there is reflection, rotation or glide reflection with regard to mirror lines or axes.

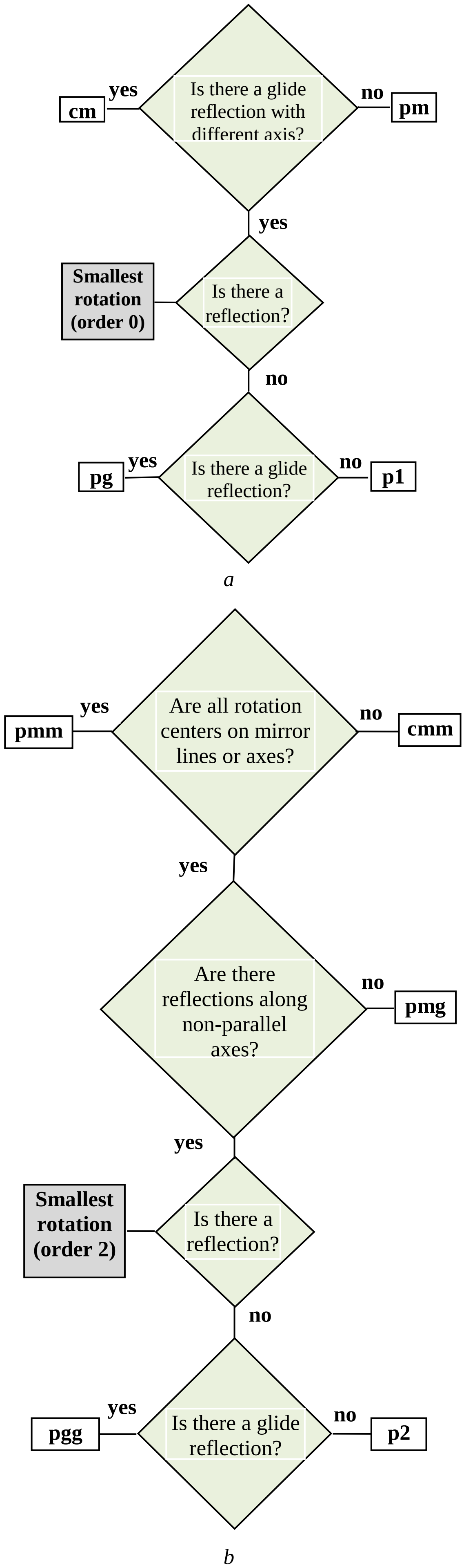


Fig. 12 Determining a pattern type of order 0 and 2

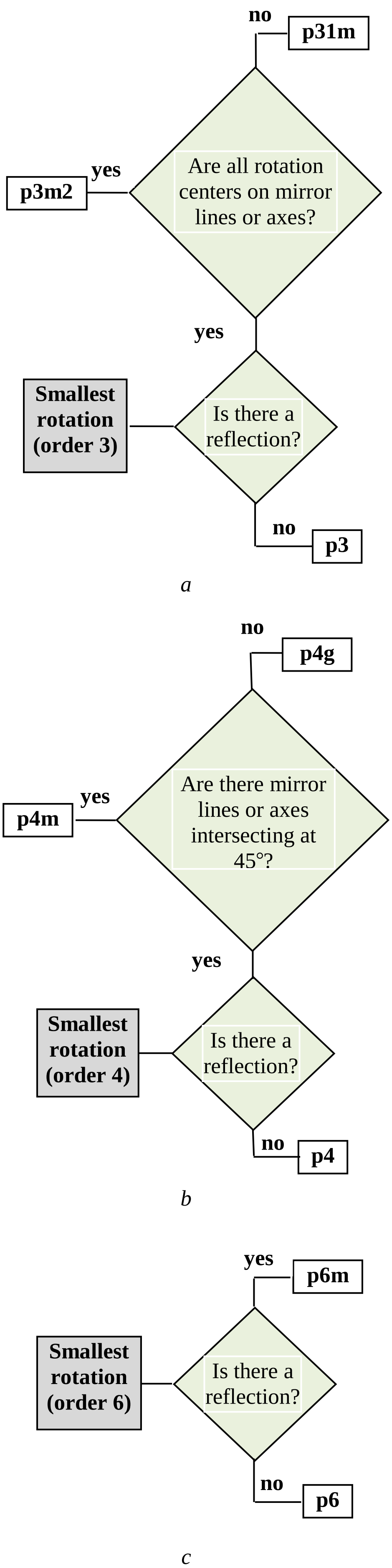


Fig. 13 Determining a pattern type of order 3, 4 and 6

In two-dimensional geometry the main kinds of symmetry are with respect to the basic Euclidean plane isometries (stretching and distortions are not allowed): translations, rotations, reflections and glide reflections.

The concept of symmetry is applied to the design of objects of all shapes and sizes. Symmetry is a crucial organizing principle of shape.

5. CONCLUSION

We are surrounded by symmetric objects and patterns. Many shapes and geometrical models exhibit symmetries: isometric transforms that leave the shape globally unchanged. From architecture to manufacturing, symmetry is the rule and many people see it as being an aspect of beauty.

Symmetry is a very important concept in mathematics and can be applied in many different areas including equations, shapes, work pieces and aero dynamical buildings [13]. Symmetry also plays an important role in human visual perception and aesthetics. Symmetry refers to the qualities of balance, proportion, shape and classical aesthetics [9].

It was seen above that motifs can be either symmetrical or asymmetrical and that symmetrical motifs can have either rotational and/or reflection characteristics. Also, border patterns can be grouped into one of seven types and plane patterns into one of seventeen types. Patterns can be described as a procession of repetition of subparts.

In any human endeavour for which an impressive visual result is part of the desired objective, symmetries of repeated patterns play a profound role.

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