Scientific Bulletin of Politehnica University Timisoara

Transactions on HYDROTEHNICS

Volume 67(81), Issue 1, 2022

THE RELIABILITY OF THE LINE-STRENGTH METHOD FOR MODELLING PARTIALLY PENETRATING WELL

David Ioan[1] Ştefănescu Camelia[1] Visescu Mircea[1]

Abstract: The modelling of the three-dimensional flow generated by a partially penetrated well/borehole into the aquifer requires the correct mathematical representation of the interdependence between the piezometric head and the water inflow/outflow along the borehole, considering the fact that this dependence, from a mathematical point of view, has singular character. This can be explained through the small diameter of the borehole/well compared to the extended flow domain dimensions and to the length of the well. To represent this singular nature of the flow, especially in the well neighboured areas, mathematical modelling methods use singularity distributions along the well axis. On this basis, several mathematical methods have been developed, starting with the derivation of some formulas for calculating the global flow rate of the partially penetrating well (e.g., the Muskat formula), up to complex mathematical methods such as the method of analytical elements. In general, these methods only allow for the calculation of the total flow yield of the partially penetrated well into the aquifer, but in numerous technical applications, knowing the water inflow velocity into the well is of particular interest. A representative technical application in this regard is the protection of the well against the entrainment of sand in the well, a phenomenon that occurs if the water inflow velocity exceeds certain limit values. Therefore, knowledge of the water inflow rate distribution along the well is of great importance.

The paper presents a calculation method based on the mathematical representation of the groundwater flow generated of partially penetrating well using of line sink segments of singularities distributed along the well axis. The proposed method allows the calculation of both the total flow yield and the distribution of the inlet flow along the well.

Keywords: partially penetrating wells, inflow rate distribution, line sink singularity segments of constant strength

1. INTRODUCTION

Groundwater wellbore like fully or partially penetrating wells are used in engineering mainly, as groundwater supply plants for drinking water, as petroleum and geothermal exploitation as coupled injection/recovery wells and so on. The main parameters of wells are the well radius, the penetration degree into the groundwater layer, the recharge/discharge rate and its distribution along the well wall and the hydraulic head. A correct groundwater flow modelling of such flows requires the consideration of the real relation between the discharge/recharge rate distribution along the well and the hydraulic head which strongly depends on well radius. This relation has singular behavior such as logarithmic singularity or polar singularity. This fact can be explained through the specific geometric features of wellbores, namely their large axial extension in comparison to its diameter which is also small in comparison to the flow domain extension. Considering this property several calculation methods are developed which assume that the groundwater flow to partially penetrated well (pW) can be represented mathematical as 3 D *line sink singularity distributed* along the well axis. Using this procedure Muskat developed formula for calculating the global discharge rate of the partially penetrated well [1], [2]. Complex mathematical methods such as the analytical elements method (AEM) can be followed in [3], [4], [5], [6], [7], [9], [10]. These methods use 2D line sink strength distribution along the external boundary of the flow domain and 3D line strength density distribution along the partially penetrating well axis.

The analytical element method (AEM) developed mostly by Strack and Haitjema [2], [3], [4]) is based on the line sink strength distribution for modelling ground-water flow in shallow regional aquifer incorporating line drainage objects like horizontal or vertical wells. Several results for modelling radial collector well with laterals based on the coupling of AEM and boundary element method (BEM) were obtained in [5], [6], [7], [8], [9], [10] using the same procedure line sink distribution.

In general, these methods only allow the calculation of the total flow yield of the partially penetrated well (pW), but in numerous technical applications, knowing the water inflow velocity into the well is of particular interest. Such a representative technical case is the sanding danger of the well, a phenomenon that occurs if the water inflow velocity in to the well exceeds certain limit values. To solve such a problem the knowledge of the water input distribution along the well is required.

The paper presents a calculation method based also on the mathematical representation of the flow using line sink segments of constant strength distributed along the well axis. The proposed method allows the calculation of both the total flow yield and the distribution of the inlet flow along the well.

2. CALCULTION THE TOTAL YIELD OF PARTIALLY PENETRATING WELL

As described above, the calculation of pW involves the determination of the total flow yield as well as the flow distribution along pW.

[1] Politechnica University Timisoara, Department of Hydrotechnical Engineering, George Enescu Street, no.1A, Zip code 300022, Timisoara, Romania, e-mail: ioan.david@gmx.net

The best-known calculation method of the total yield of pW was proposed by Muakat [1]. In Figure 1 can be sea the calculus scheme and the most important notation of a pW .

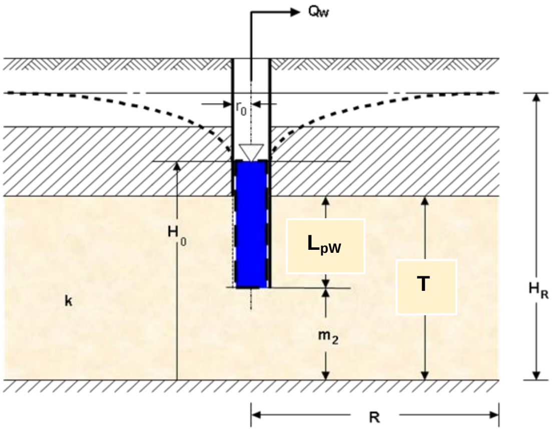


Figure 1 Scheme of a partially penetrating well

The groundwater flow is a 3D flow close to pW end and predominantly 2 D flow outside the pW close range. Consequently, solving the groundwater flow problems generated by pW requires solving a boundary value problem for the 3D Laplace equation considering the characteristic boundary conditions, i.e., a confined aquifer of thickness **m** which above and below is limited by impermeable layers and given piezometer head $\mathbf{H_R}$ at the influence radius of **R** and given piezometer head $\mathbf{H_0}$ at the well screen $\mathbf{r_0}$ (Figure 1).

Using the basic singularity solution of 3D Laplace equation distributed along of pW Muscat obtains the following calculation formula for the total yield Q_{pW} of the pW:

$$Q_{pW} = \frac{2\pi k T\left(H_R - H_0\right)}{ln\frac{R}{r_0} + \left(\frac{T}{L_{pW}} - 1\right) ln\frac{4T}{r_0} - \frac{T}{2L_{pW}}\left(f\left(\frac{L_{pW}}{T}\right)\right)} \qquad (1)$$

The function $f(m_1/m)$ is given as

$$f\left(\frac{L_{pW}}{T}\right) = ln\frac{\Gamma(0.875\frac{L_{pW}}{T})\cdot\Gamma(0.125\frac{L_{pW}}{T})}{\Gamma(1-0.875\frac{L_{pW}}{T})\cdot\Gamma(1-0.125\frac{L_{pW}}{T})} \qquad (2)$$

where

Γ is the Gama-function.

For technical calculations can be use the function values given in the Table 1.

Table 1. The $f(m_1/m)$ function values in terms of m1/m values

Muscat formula and its derivatives was discussed detailed by Frank [11], establishing that the Muskat formula for the total yield of partially penetrating well (pW) can be successfully applied pW if the ratio between the well radius and the aquifer is less than 0.1 (i.e., $r_{pW}/m < 0.1$).

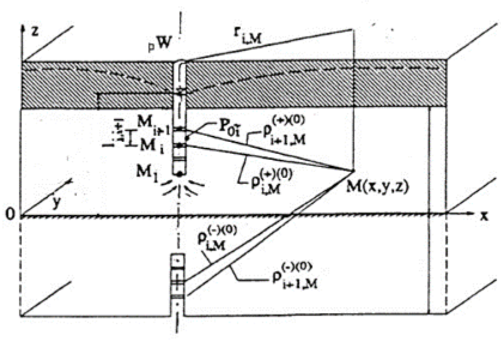
Likewise, the Muskat formula is useful to prove the reliability of grid discretization based numerical methods, being known as these methods leads to difficulties in neighborhood of singular flow conditions like well.

Unfortunately, the Muskat formula does not allow the calculation of the inflow rate distribution along the pWell, however, knowing this distribution is important for solving some dimensioning and exploitation problems of pW such as: determination of the pW segments with increased inflow velocity to prevent sand entering in the borehole such as the portion at the end of pW; coupling \the internal flow in the pipe with the external groundwater flow to pW. In the next paragraph an approach for determining the inflow rate distribution along pW will be presented.

3. CALCULATION METHOD FOR INFLOW RATE DISTRIBITION ALOG OF PARTIALLY PENETRATING WELL

The possibility to determine the inflow rate distribution in pW is offered by numerical methods. The grid discretization-based methods are generally not suitable because can lead to difficulties in calculating the inflow rate distribution, caused by the singularity behavior of the groundwater flow close to pW. In this regard in [12] the reliability of the most used standard numerical modeling software of groundwater flow PMWIN/MODFLOW (Processing MODFLOW for Window) was analyzed. The performed numerical experiment for partially penetrating wells (pW) using different discretization sizes showed that numerical methods like MODFLOW can lead to considerable errors regarding the inflow rate distribution along the pW. In [12], an extension of the well index-based method is proposed and tested, which allows the correction of the numerical calculated inflow discharge rate distribution along the pW.

An acceptable method to solve the inflow rate distribution along the pW well can offer the line sink singularity distribution along the pW axis which is proposed and tested in the present paper. As basic mathematical representation of the groundwater flow generated by pW the potential function of a number of N_ψ line sinks of constant strength ($\psi_{i,i+1}$, $i=1,2,\ldots, N_\psi$) distributed along the pW axis will be considered. To comply with the impermeability boundary conditions at the lower and upper part of the aquifer, the mirroring technique (i.e., the method of images) of the basic modules sketched in Figure 2. will be applied.



1.0
0.0

Figure 2 Discretization of pW and his mirroring relative to ox axis as basics module for flow potential determination

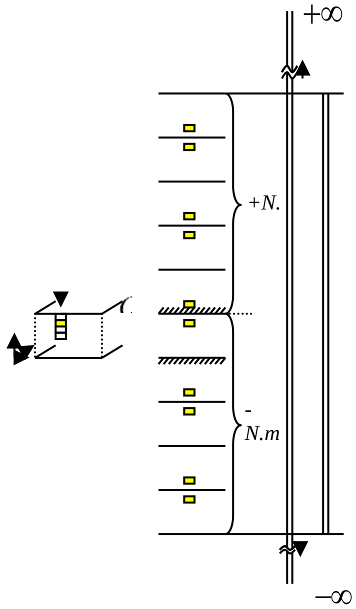


Figure 3. Sketch of the +/-N mirroring bands of pW elements

The method of images consists in (+/-N times) mirroring of the basic module sketched in Figure 3. The effect of the domain situated outside of the +/-N mirroring can be modeled as semi-infinite line-sink. This effect is relatively small for N>3 and will be neglected.

Using the notations in Figure 2 the mathematical expression of the potential function in any point M(x,y,z) of the flow domain is expressed as [7]:

$$\phi_{pW}(M) = -\sum_{i=1}^{N_\psi} \psi_i \, G_i\left(M, pW\right)$$

$$\text{with} \tag{3}$$

$$G_i\left(M, pW\right) = \frac{T}{2\pi}\sum_{n=0}^{N} ln\left[R_{i,M}^{(+)(n)} \cdot R_{i,M}^{(-)(n)} \cdot R_{i,M}^{(+)(-n)} \cdot R_{i,M}^{(-)(-n)}\right]$$

The used notations in (3) have the following helpful expressions:

$$R_{i,M}^{(\pm)(\pm n)} = \frac{\rho_{i,M}^{(\pm)(\pm n)} + \rho_{i+1,M}^{(\pm)(\pm n)} - l_{i,i+1}}{\rho_{i,M}^{(\pm)(\pm n)} + \rho_{i+1,M}^{(\pm)(\pm n)} + l_{i,i+1}}$$

$$\tag{4}$$

$$\rho_{i,M}^{(\pm)(\pm n)} = \sqrt{\left[(x - x_{M_i})^2 + (y - y_{M_i})^2 + (\pm 2\, n\, m \pm z - z_{M_i})^2\right]}$$

$$\rho_{i+1,M}^{(\pm)(\pm n)} = \sqrt{\left[(x - x_{M_{i+1}})^2 + (y - y_{M_{+1i}})^2 + (\pm 2\, n\, m \pm z - z_{M_{i+1}})^2\right]}$$

The $\boldsymbol{N_\psi}$ unknow line strength densities $\boldsymbol{\psi_i}$ can be calculated using the potential function (3) and the boundary conditions similarly to Muskat formula application presented above: confined aquifer of thickness **T** which above and below is limited by impermeable layers and given piezometer head $\mathbf{H_R}$ at the influence radius of **R** and given piezometer head $\mathbf{H_0}$ at the well screen $\mathbf{r_{pW}}$ (Figure 1). *On the well screen only one point for each line sink (i.e, N_ψ points) can be considered.*

The question is how can be calculate the inflow rate distribution into pW knowing the line sinks strength ψ_i, i.e., the solutions of the above formulated boundary value problem.

As for the *total inflow yield of the pW* the following formula is valid:

$$Q_{pW} = \sum_{i=1}^{N_\psi} \psi_{i,i+1} \cdot l_{i,i+1} \tag{5}$$

To prove the reliability of the proposed approach for calculating of the total yield of pW an example with the following par ameters is considered: groundwater layer thickness **T = 10,00 m**; pW penetration L_{pW}**=5m; $\mathbf{k_f}$=10 4m/**s; the considered line sink elements along the pW axis are N_ψ=5, N_ψ=15 and N_ψ=50; well radii r_{pW}= 0,25m, 0,5m and 1m respectively.

In Table 2 are presented comparative results for total yield calculated with the proposed approach (5) and the Muscat formula (3). The results show that of the total yield of pW calculated with the proposed method is already reliable for relatively small number of pW elements (N_{pWE}>3).

Table 2 comparative results for total yield of pW

r_{pW} (m)		**0,25**	**0,50**	**1,00**
(r_{pW}/L_{pW})		0.05	**0.10**	**0.20**
Muskat [1]		**1,70**	**2,10**	**2,73**
Proposed approach	**N_ψ = 3**	1,65 (3%)	2,00 (5%)	2,55 (7%)
	N_ψ = 5	1,66 (2%)	2,03 (2%)	2,60 (3%)
	N_ψ = 10	1,67 (2%)	2,06 (2%)	2,65 (3%)
	N_ψ = 15	1,69 (<1%)	2,09 (<1%)	2,70 (1%)
	N_ψ = 30	1,70 (0.0%)	2,11 (<1%)	2,75 (<1%)
Parameters	T = 10,00 m; k_f=10^{-4} m/s; R=50m; S=2m; H_R-H_0=2m, L_{pW} = 5,00m			

These results allow establishing the conditions in which the proposed method is reliable and applicable both for the calculation of the total flow of the pW and of the distribution of the flow along the pW. Analysing the results presented in Table 2 we can see that for small radii relative to pW length i.e., $r_{pW}/L_{pW} \leq 0.05$ the proposed method can already be applied considering relatively small number of line sinks namely $N_\psi \geq 3$ if we accept a maximum error of 3%. This minimum value of line sinks can be extended also for relatively high radii of pW like $r_{pW}/L_{pW} \leq 0.20$.

Further, the *extension and application possibilities of the proposed line sinks distribution-based method* to *determine the inflow rate distribution along of pW* will be analysed. In this regard a discretization of the pW in N_{pWE} elements of length L_{WE} will be considered. It is obvious that, $N_\psi \geq N_{pWE}$, that is, the number of line sink segments (N_ψ) must be greater than the number of discretization elements of pW (N_{pWE}). For the limit case when $N_\psi = N_{pWE}$ the inflow rate on pW-Element ($q_{pWEi,i+1}$) would be equal to line sink strength ($\psi_{i,i+1}$):

$$q_{pWEi,i+1} = \psi_{i,i+1} \qquad (6)$$

To test this statement in Figures 4 and 5 the $\psi_{i,i+1}$ distributions are shown. The results of the performed calculus presented in Figure 5 and Figure 6 namely the dependency of the line strength $\psi_{pi,i+1}$ distribution on the quotient r_{pW}/L_{WE} (well radius relative to the length of discretisation well element) shows that the line sink strength distribution for larger radii (e.g., r_{pW}=0,5m and r_{pW}=1.00m) becomes oscillatory even negative values. Such a distribution of inflow rate is not realistic. Therefore, the assumption (6) only for small values of the ratio between the well radius and the length of the pW-Elements i. e. r_{pW}/L_{WE} <0.25 make sense. But this is a qualitative result only.

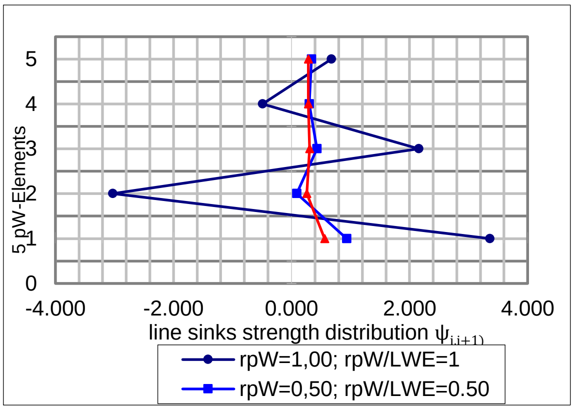


Figure 4 Distribution of line sink strength density distribution for a discretization of five well-elements (N_{pWE}=5)

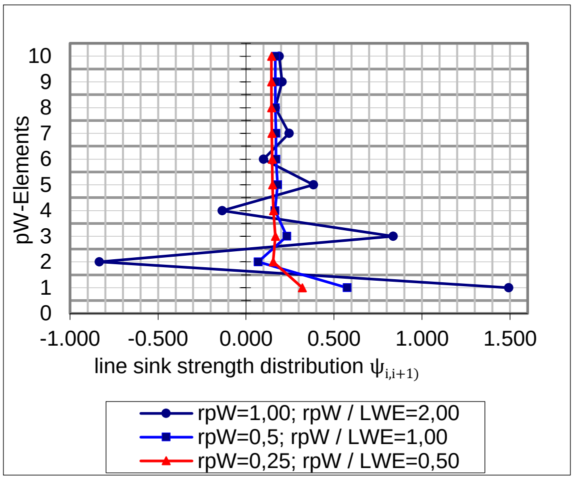


Figure 5 Distribution of line sink strength density distribution for a discretization of ten well-elements (N_{pWE}=10)

Based on the results for the total pW yield calculation presented in paragraph 2, namely, that the total pW yield is relative correct if the number of line sinks is at least 3 (i.e., $N\psi \geq 3$) a transmission/extension of the proposed method for calculation of the inflow rate distribution along the partially penetrating well (pW) can be obtained. Due to these aspects the following calculus procedure for inflow rate distribution along a pW is proposed:

- Discretisation of pW in N_{pWE} Well-Elements of length L_{WE}
- Choosing a number $N_{\psi WE} \geq 3$ of line sinks for each element (e.g. 3, 6,10,…)
- calculate the $\mathbf{N_\psi}$ = N_{pWE} x $N_{\psi WE}$ unknow line strength densities $\boldsymbol{\psi_i}$ using the potential function (3) and the boundary conditions similarly to Muskat formula and to example presented above

To prove and to demonstrate the reliability of the proposed method and calculus procedure the same examples as above are discussed considering N_{pWE}=5 well-elements with lengths of L_{WE}=1m and a number of $\boldsymbol{N_\psi=3}$-line sinks for each well-element. In figures 6 and 7 are shown comparative results of calculated inflow rate distribution considering valid the equality (5) i.e., the assumption that the inflow rate coincides with line-sink strength (named pervious results) and the calculated inflow rate distribution calculated using the proposed method considering 3 line-sinks for each well-element (named new results).

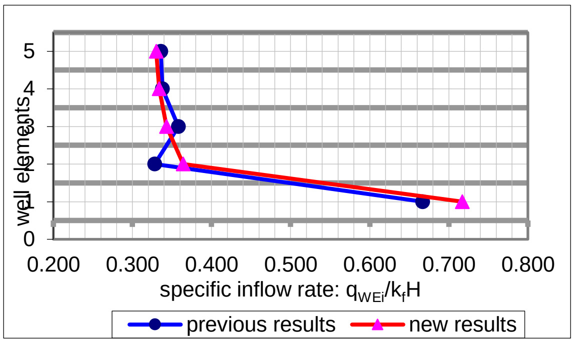


Figure 6. Distribution of line sink strength density distribution (pervious results) and of inflow rate distribution (proposed method) for a well radius of r_{pW}=0.5m

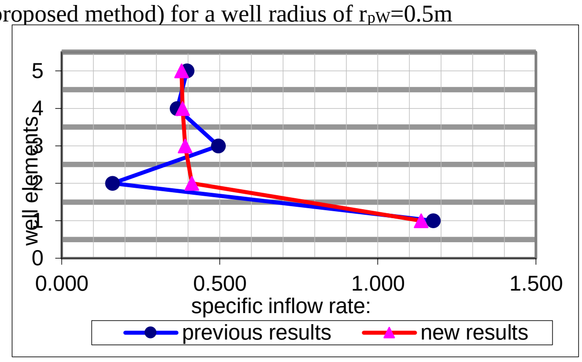


Figure 7. Distribution of line sink strength density distribution (pervious results) and of inflow rate distribution (proposed method) for well radius of r_{pW}=1.0 m

The calculations for the example considered above were also performed for N_ψ =6 per W-Element which ensures a reduction of errors (see table 2). The obtained results presented in Table 3 show that the differences between the specific inflow distribution calculated with N_ψ =3 line-sinks and N_ψ =6 line-sinks per well-element are relative small with little meaning from a technical point of view. Larger difference occurs only for the Element at the end of the pW.

Table 3. Inflow rate distribution alon the pW

Well Element	r_{pW}/L_{WE} =0.5			r_{pW}/L_{WE} =1.0		
	N_{ψ} per Well-Element			N_{ψ} per Well-Element		
	1	3	6	1	3	6
1	*0.667*	0.717	**0.755**	*1.176*	1.137	**1.231**
2	*0.328*	0.364	**0.358**	*0.161*	0.412	**0.400**
3	*0.358*	0.343	**0.339**	*0.496*	0.391	**0.385**
4	*0.338*	0.334	**0.331**	*0.365*	0.383	**0.377**
5	*0.336*	0.330	**0.328**	*0.397*	0.379	**0.374**
Total yield ∑	**2.03**	**2.09**	**2.11**	**2.60**	**2.70**	**2.75**
Total yield Muskat	**2.10**			**2.73**		

It can be seen also that the total yield value is practically the same for 3 and 6 line-sinks per well element.

4. CONCLUSIONS

In the paper is proposed and tested a calculation method of groundwater flow generated of partially penetrating well using of line- sink singularities distributed along the well axis. The proposed method allows the calculation of both the total flow yield and the distribution of the inflow rate along the well. Several test calculation examples were performed which attests the efficiency and reliability of the method.

The proposed method can be used for testing of the reliability of discretization-based numerical methods like standard FVM (e.g., MODFLOW), being known that these numerical methods have difficulties to correct modelling the singularity flow behaviour at vicinity of well.

REFERENCES

[1] *M., Muskat*, The Flow of Homogeneous Fluids through Porous Media. McGraw-Hill, Ann Arbor, Michigan, 1937

[2] *H. M., Haitjema*, Modeling Three-Dimensional Flow in Confined Aquifers using Distributed Singularities. PhD thesis, University of Minnesota. Doctoral thesis, 1982

[3] *H. M., Haitjema*, Analytic Element Modeling of Groundwater Flow. Academic Press, New York London Toronto, 1995

[4] *O. D. L., Srack*, Groundwater Mechanics. Prentice Hall, Englewood Cliffs, New Jersey, 1989

[5] *H. M., Haitjema, V. A., Kelson and H., Luther*, Analytic Element Modeling of Groundwater Flow and High-Performance Computing. U.S. EPA/600/S-oo/oo1, May 2000

[6] *I., David*, Grundwasserfassungsanlagen mit Filterrohren. Technischer Bericht Nr. 19, TH Darmstadt, Institut für Hydraulik und Hydrologie, 1973

[7] *I., David, H., Gerdes*, Incorporation of local three-dimensional flow in plane BEM. Boundary Element XVII. Computational Mechanics Publications. Southampton Boston, 1995

[8] *I., David, H., Gerdes*, Coupling Analytical Element, BEM and FEM to develop a model for groundwater flow, Computational Mechanics Publication, computational Methods in Water Resources, Vol.1, 362-370, 1998

[9] *I., David*, Numerical method to solve groundwater problems in shallow aquifer containing artificial recharge and recovery systems. European Geophysical Society XXVI General Assembly Nice, France, 2001

[10] *I., David*, Analytical and Boundary Elements based Integral Representation for Numerical Solution of 3-D Potential Problems in Heterogeneous Media Containing Singularities, 12th WSEAS Int. Conf. On mathematical and computational methods in science and engineering (MACMESE'10), University of Algarve, Faro, Portugal, November 3-5 (invited plenary speaker), 2010

[11] *O.L., Franke*, Die Srömungsvorg#nge bei unvollkommenen Brunne, Mitteilungsblatt der Bundesveranstallt für Wasserbau, Karlsruhe, 1965

[12] *I., David at al*, Numerical Experiment to Analyse the Reliability of Coarse Grid based Numerical Methods for Modelling Groundwater Flow by Drainage Objects like Wells in Confined Aquifer, WMCAUS 2016, Procedia Engineering 161, 1949–1956, 2016