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Sections and Intersections of the Surfaces in the History of Geometry

Lucian RAICU[1] Ileana VASILESCU[1]

Abstract: This paper proposes an analysis of the concept of relationship between surfaces, exemplified by the projections, sections and intersections thereof. There are given, the first examples of these relationships: the junction of torus, cylinder and cone made of Archytas in Tarent (c. 435 BC – 347 BC), spiric sections resulted from the intersection of a torus with a plane parallel to its axis, circular right cone sections as performed by Germinal Pierre Dandelin (1794 – 1847), and circles obtained by Yvon Villarceau (1813 – 1883) by slicing a torus with a bitangent plane (diagonal plane).

Keywords: section, intersection, surface, geometry, torus, cylinder, cone, ellipse, parabola.

1. INTRODUCTION

Geometry is a component of mathematical science studying shapes and surfaces as well as many of their applications. In a strict sense, the geometry is the study of shapes and size figures. Geometry was one of the two domains of premoderne mathematics, other commanding the study of numbers.

In its historical evolution the geometry was concerned by formal essence and beauty of geometric constructions, as evidenced by the "*Philebos*" dialogue of Plato. There was the possibility for that concept of geometry to result in depth study of plane and spatial constructions which can be achieved using a compass and square (or ruler), and the three major problems of ancient geometry: squaring the circle, duplicating the cube and the angle three-section. Attempts to resolve these problems and restrictions have not impeded the development of geometry, but on the contrary they have opened new ways in the evolution of mathematics, especially to analytic geometry. In Antiquity, the science of space figures was considered to be the safest and the most intuitive of all spiritual achievements.

Archytas's attempts to resolve the problem of duplication of the cube had led to the definition of space curves, considered by some experts as the first of its kind in the history of geometry.

2. ARCHYTAS'S CURVE

Archytas in Tarent (c. 435 BC – 347 BC), ancient Greek philosopher, mathematician, astronomer, and politician was a disciple of Pythagoras and a good friend of Plato.

In geometry, he is considered to be the first to represent *the cube*, *tetrahedron* and *dodecahedron* (although according to some sources, *dodecahedron* was first described by Hippasus of Metapontum, 5th century BC).

Archytas in Tarent named the harmonic mean, important much later in projective geometry and number theory, though he did non invent it.

According to Eutocius (c. 480 – c. 540), Archytas solved the problem of cube duplication by using a space curve (reckoned by some experts as the first known historical space curve) which now forms part of the curves called *duplicatrice*. Archytas had found a three dimensional solution to solve the problem by using intersection of three geometric objects: torus, cylinder and cone (Fig. 1, Fig. 2 and Fig. 3).

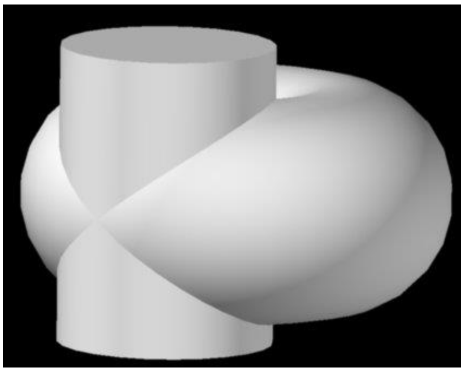

Fig. 1 Intersection of torus and cylinder, 3D

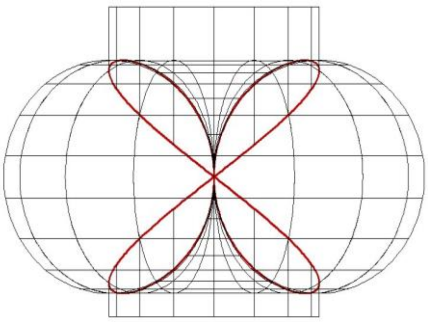

Fig. 2 Intersection of torus and cylinder,

[1]POLITEHNICA University of Bucharest, Engineering Graphics and Industrial Design Department, Splaiul Independentei nr. 313, Code 060042, Bucharest, Romania, E-mail: lucian.raicu@gmail.com; ileana.vasilescu@gmail.com

Space curve resulting from this intersection is called *a Archytas's curve* (Fig. 3).

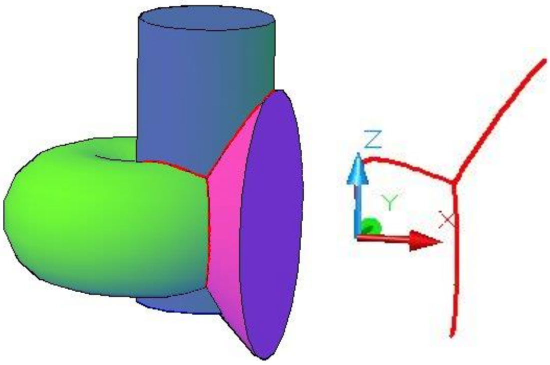

Fig. 3 Archytas's curve

The result obtained by Archytas in Tarent appears to us as extraordinary, given the solution obtained synthetically, without using the Cartesian coordinate system.

3. SPIRIC SECTIONS

Spiric sections (from the ancient Greek word σπειρα which means torus). Historical, these are curves defined as intersections of a torus with a plane parallel to its axis (Fig. 4 and Fig. 6). Spiric sections were first made by the Greek geometer Perseus (c. 150 BC), according to the mentions of Geminus of Rhodes (1st century BC) and of Proclus (412 – 485). Among the spiric curves you can mention: *hipopeda* studied by Eudoxus (c. 408 BC – c. 347 BC), Proclus, and J. Booth (1810-1878), *Cassini's ovals*.

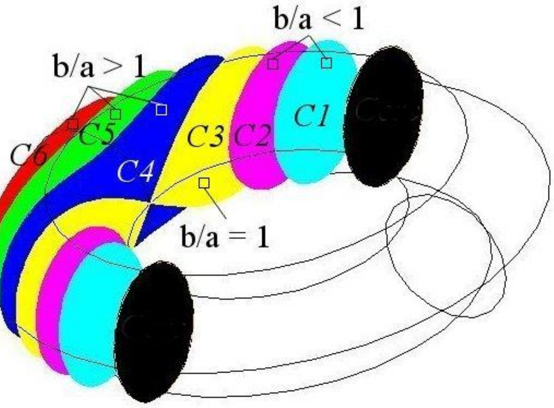

Fig. 4 Spiric sections (spiric curves), 3D representation

Cassini's ovals (Fig. 5) are the *spiric curves* studied by the Italian astronomer Giovanni Domenico Cassini (1625 – 1712).

These curves are defined as the locus of points in the plane for which the product of distances to two fixed points called the focus *F1* and *F2* is constant.

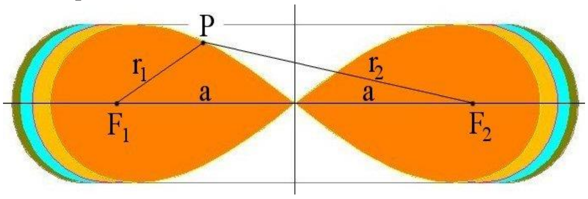

Fig. 5 Cassini's ovals

In a frame of reference (xOy) these curves (Fig. 5) satisfy the relation:

$$dist(F_1, P) \times dist(F_2, P) = b^2 \text{ sau } r_1 r_2 = b^2 \qquad (1)$$

where *b* is a constant.

Noting with *a* focal distances (Fig. 5) it yields:

$$[(x-a)^2 + y^2] \cdot [(x+a)^2 + y^2] = b^4 \qquad (2)$$

or:

$$(x^2 + y^2 + a^2)^2 - 4a^2x^2 = b^4 \qquad (3)$$

This is the Cartesian default equation of *Cassini's ovals*.

The shape of these curves depends on the ratio *b/a*, as follows:

- *b/a* < 1, the locus is a set of two non-secant curves (Fig. 4, *C1* and *C2*; Fig.6; Fig. 6, *C1* and *C2*);
- *b/a* = 1, the locus is a curve named *lemniscate* (Fig. 4, *C3*; Fig. 6, *C3*). This curve, *infinity* symbol-shaped ∞ was studied by the mathematician Jacques (Jakob) Bernoulli (1654 – 1705), hence the name of *the lemniscate of Bernoulli*;
- *b/a* > 1, the locus is a *continuous curve* which can be: with four points of inflection (Fig. 4, *C4*; Fig. 6, *C4*) or like an ellipse (Fig. 6, *C5* and *C6*).

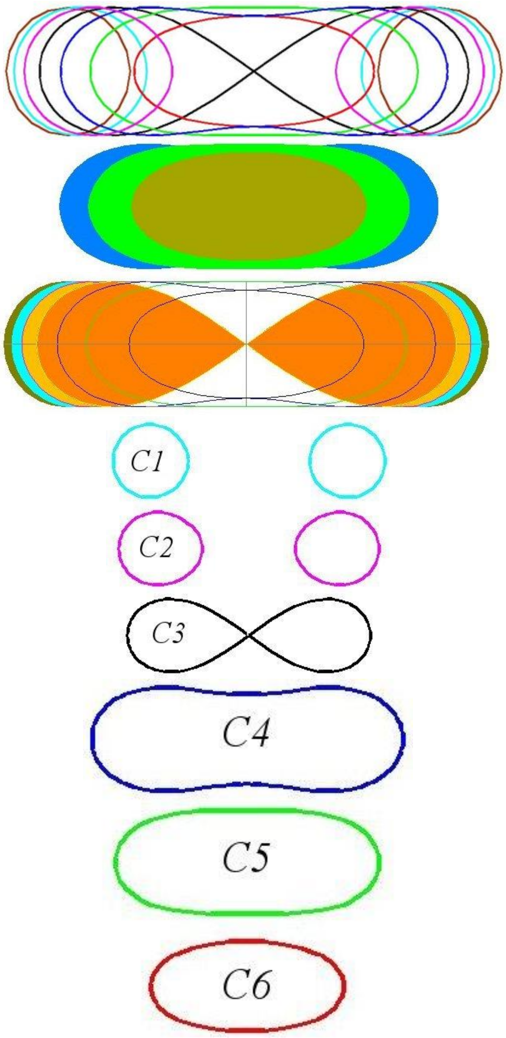

Fig. 6 Spiric curves

4. CONIC SECTIONS

Conic section (or conic) is a curve obtained as the intersection of a cone (a right circular conical surface) with a plane. In geometry, a conic may be defined as a plane curve of degree 2.

The three types of conics are the *ellipse*, *parabola*, and *hyperbola* (Fig. 7 and Fig. 9). The circle (Fig. 8 and Fig. 9) is a special case of the ellipse. It can be considered as a fourth type (as it was by Apollonius).
The circle and the ellipse arise when the intersection of cone and plane is a closed curve.

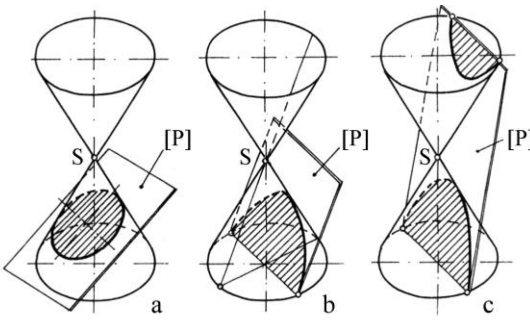

Fig. 7 Conic sections, ellipse, parabola, and hypebola

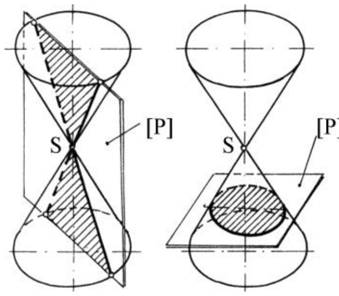

Fig. 8 Section triangle and section circle

The conic sections were named and studied at least since 200 BC, when Apollonius of Perga (c. 262 BC – c. 190 BC) undertook a systematic study of their proprieties. It is believed that the first definition of a conic section is due to Menaechmus (380 BC – 320 BC), Greek geometer.

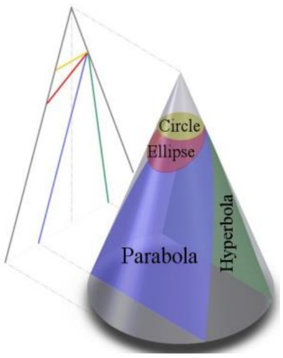

Fig. 9 Types of conic sections

Germinal Pierre Dandelin (1794 – 1847), Belgian mathematician is known for his work in geometry and in particular on conical figures. Among his results we can note theorem of Dandelin proved in 1822.

Theorem of Dandelin. *Flat section in a rotating cone is an ellipse, parabola, or hyperbola as the plane passes through the apex of the cone and it is parallel to the secant, not section the cone, tangent to the cone, the cone or it cuts off after two generators. Or: Section carried out with a plan in a right circular cone is an ellipse* (Fig. 7 a, Fig. 10 and Fig. 11), *parabola* (Fig. 7 b, Fig. 10 and Fig. 12) *or hyperbola* (Fig. 7 c), *as the section plan cuts a single canvas of the cone,it is parallel to a tangent plane to the cone or cut both canvases of the cone*.

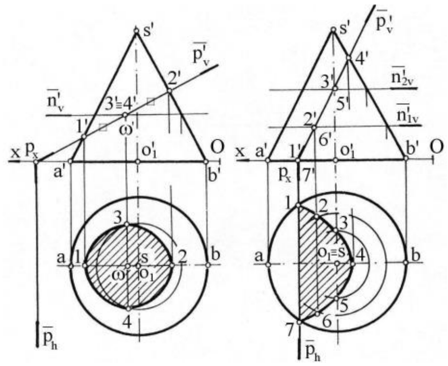
Fig. 10 Conic sections: ellipse and parabola

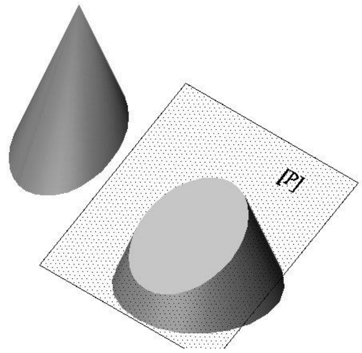

Fig. 11 Conic section, ellipse (3D representation)

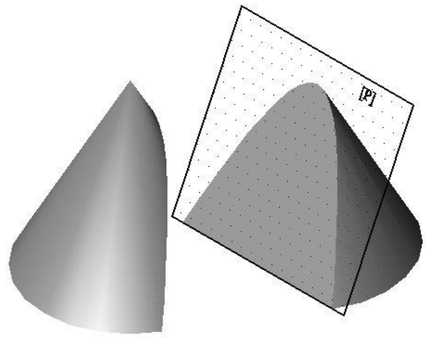

Fig. 12 Conic section, parabola (3D representation)

5. CIRCLES OF VILLARCEAU

Circles of Villarceau are two circles obtained by cutting a torus with a diagonal plane (end bitangent plane) which runs through the center of the torus (Fig. 13, Fig. 14 and Fig. 15). These circles are named after the French astronomer and mathematician Yvon Villarceau (1813 – 1883).

Through a point on the torus we can build four circles: one in a plane parallel to the equatorial plane, the other in a plane perpendicular to it, and the last two are Villarceau's circles, equal among themselves, whose diameter is equal to the diameter of the limit circles. To get those two circles of Villarceau, the torus of the equation:

$$(x^2 + y^2 + z^2 + R^2 - r^2)^2 = 4R^2\,(x^2 + y^2) \qquad (4)$$

should be sectioned with an plan by the equation:

$$\mathrm{k}x = \mathrm{m}z. \qquad (5)$$

Application: In order to obtain the circles of Villarceau, the torus with r = 3 (the radius of the generator circle) and R = 5 (the radius of the limit circle), of the equation:

$$(x^2 + y^2 + z^2 + 16)^2 = 100(x^2 + y^2) \qquad (6)$$

should be sectioned with an plan by the equation:

$$3x - 4z = 0. \qquad (7)$$

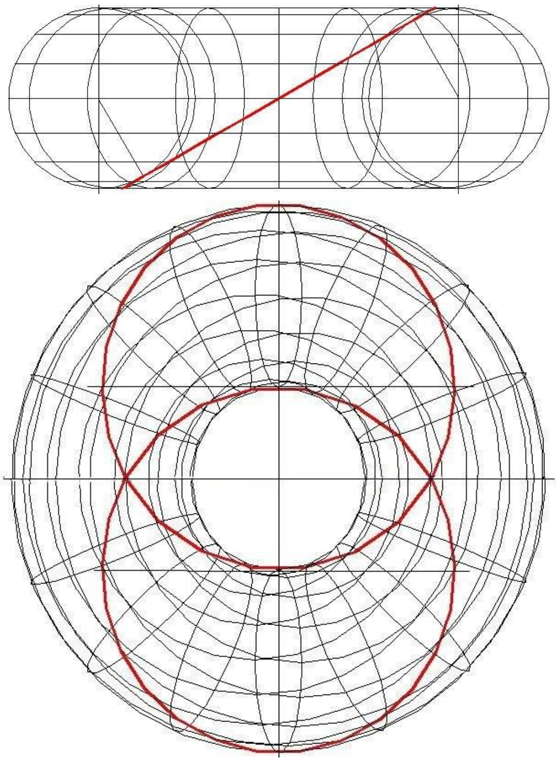

Fig. 13 Circles of Villarceau, orthogonal projection

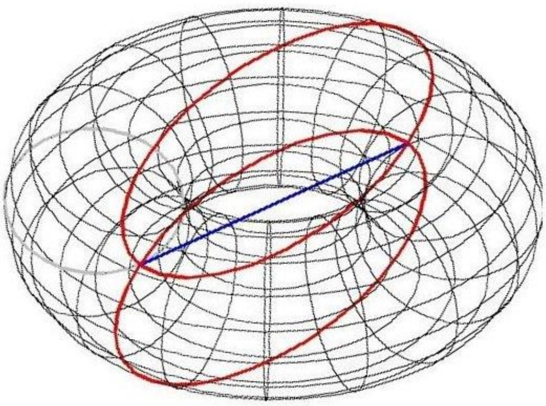

Fig. 14 Circles of Villarceau (3D representation)

In architecture, we meet a beautiful composition with a torus and circles of Villarceau at *Musée de L'Oeuvre Notre-Dame*, Strasbourg, 16th century (Fig. 15).

Fig. 15 Circles of Villarceau in architecture

6. CONCLUSIONS

The study of spatial geometrical shapes and of the relationship between them has been a constant preoccupation of mathematicians throughout the geometry history. Many of the results achieved have found numerous applications in various fields of science, technology, art, architecture and design, through the information that was available and consisting in plane and spatial figures, with their qualitative and quantitative properties, in position relationship and transformations between them.

Computer use has led to the development of multiple facilities in representations of some spatial structures and the aided design, through the use of specialized programs, has led to the growth of the "quantity" of realism that is contained in the drawings made with the help of this tool.

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