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Issues on Ruled Conoid-Type Surfaces

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Abstract: Less common than the cylindrical, conical and polyhedral surfaces, conoid surfaces have started to be applied in constructions at the same time with the appearance of new materials and modern processes. Their study is important especially when we wish to create extraordinary esthetic forms. This paper presents the way in which such surfaces are generated, giving examples for right conoids. The paper uses the following methods: mathematics, epure construction and computer generation. It studies the intersection with various planes and in the end, it presents the most important applications.

Keywords: conoid, conoid surface, ruled surface, plane section

1. INTRODUCTION

Straight lines subjected to some geometric conditions generate ruled surfaces. The conoid surfaces, on which the paper refers to, are ruled surfaces, for which the generatrix is parallel to a director plan, intersects a fixed line called line of striction or the conoid axis and rests on a given curve, called the directrix or the base curve. Charles Tinseau first gave this definition in 1870.

There are several types of conoids. According to the position of the axis in relation to the director plane, it can be right (the axis is perpendicular on the director plane) or oblique (the axis has a random position). In addition, the shape of the base curve (circle, parabola, ellipsis) produces various shapes of surfaces. In addition, we find special conoids which bear the name of their researchers: Plüker, Zindler, Küper, Wallis, Gaudi [4], [6] etc.

2. CALCULATING THE CONOID SURFACES IN ANALITICAL GEOMETRY. PARTICULAR CASES

This paper operates with the following notations: **Π**– director plane, **Γ**– curved-line directrix, **Δ**– fixed line, **G**– generatrix and **Σ**– obtained surface. According to [1] and [5], take plane **Π** with the equation:

$$A1x + B1y + C1z + D1 = 0 \qquad (1)$$

line **Δ** given as the intersection line for two planes P2 and P3:

$$\begin{cases} A2x + B2y + C2z + D2 = 0 \\ A3x + B3y + C3z + D3 = 0 \end{cases} \qquad (2)$$

and the directrix **Γ** as the intersection of two surfaces:

$$\begin{cases} F1\left(x,y,z\right) = 0 \\ F2\left(x,y,z\right) = 0 \end{cases} \qquad (3)$$

Mentioning the fact that the generatrix is situated in a mobile plane that is parallel to the director plane and at the same time in a variable plane that passes through **Δ**, results the system:

$$\begin{cases} \Pi = \alpha \\ P2 = \beta P3 \end{cases} \qquad (4)$$

In addition, the generatrix rests permanently on the base curve, $G \cap \Gamma \neq 0$, hence the relation:

$$\Sigma\left(\alpha,\beta\right)=0 \qquad (5)$$

By eliminating the parameters α and β between the equations (3) and (5), we obtain the mathematical expression of the conoid surface:

$$\Sigma\left(\Pi, \frac{P2}{P3}\right) = 0 \qquad (6)$$

We consider further a few particular cases:

a) If the director plane is the horizontal projection plane, the line of striction is the Oz axis and the directrix is a circle situated in a frontal plane, the equation of surface will be obtained from the system:

$$\begin{cases} z = \alpha, \qquad \dfrac{y}{x} = \beta \\ y = b \\ \left(x-a\right)^2 + \left(z-c\right)^2 - R^2 = 0 \end{cases} \qquad (7)$$

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where a, b and c are the coordinates of the director circle center with radius R, according to figure 1.

By replacing the coordinates x, y, z from the first equations to the last we obtain:

$$\left(\frac{b}{\beta}-a\right)^2+(\alpha-c)^2-R^2=0 \qquad (8)$$

Eliminating then α and β, the equation of this conoid (also called conical edge or Wallis's conoid) is:

$$\left(b\frac{x}{y}-a\right)^2+(z-c)^2-R^2=0 \qquad (9)$$

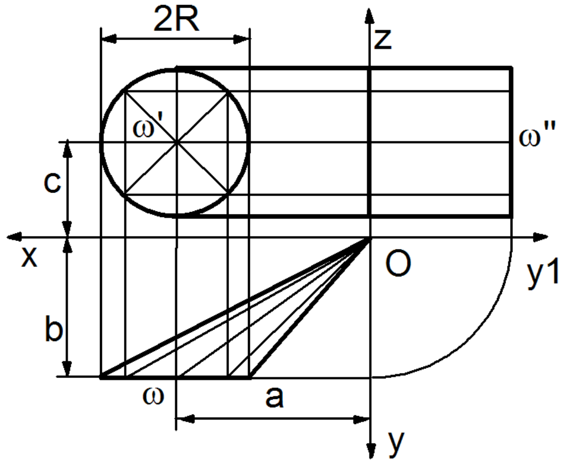

Fig. 1 Conoid with circular directrix in the frontal plane

b) If the directrix is a parabola situated in the same frontal plane, the equation for the surface becomes:

$$\left(b\,\frac{x}{y}-a\right)^2-2\,p\,(c-z)=0 \qquad (10)$$

where a, b and c are the coordinates of the vertex of the parabola, according to figure 2.

c) When the director plane is the vertical projection plane, the line of striction - the axis Oy and the directrix is a circle situated in a profile plane (fig.3), the equation for the surface will be obtained from the system (11):

$$\begin{cases} y=\alpha, & \dfrac{z}{x}=\beta \\ x=a \\ (y-b)^2+(z-c)^2-R^2=0 \end{cases} \qquad (11)$$

and that is:

$$(y-b)^2+\left(\frac{a\,z}{x}-c\right)^2-R^2=0 \qquad (12$$

3. THE EPURE REPRESENTATION OF THE RIGHT CONOID

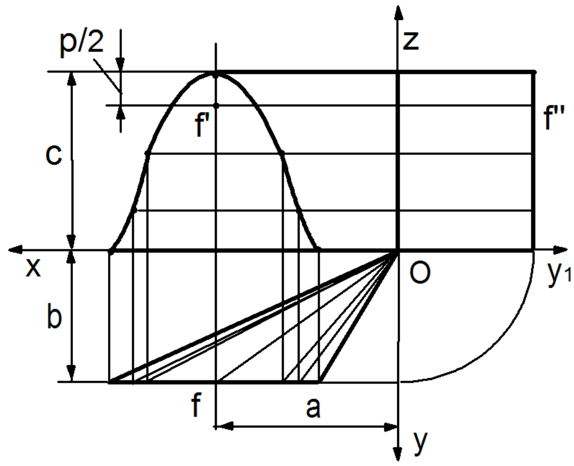

Fig. 2 Conoid with directrix-parabola in the frontal plane

The constructive variants from chapter 2 can be easily represented in epure.

a) For the conoid with the circular directrix in frontal plane and director plane xOy, the generatrixes are the horizontal lines drawn through the points of equal division of the circle with center Ω (ω, ω'). The tangents to the circle and the Oz axis give the vertical apparent outline (fig. 1).

In the horizontal plane, the generatrixes are defined by the origin and the projections of division points of the director circle.

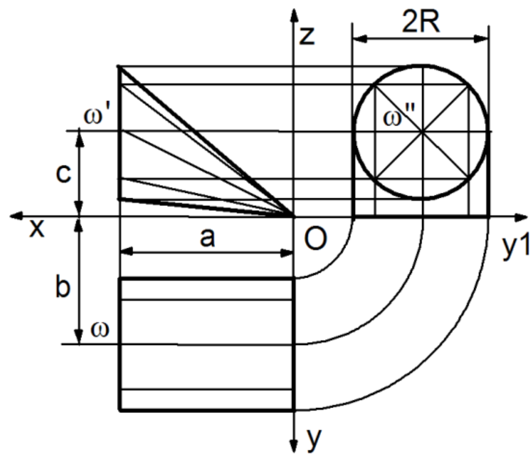

Fig. 3 Conoid with circular directrix in the profil plane

b) The construction of the epure is similarly for the conoid with the base curve - a parabola in frontal plane (fig. 2).

c) For the conoid with the circular directrix in profile plane and the xOz - the director plane, the generatrix are the frontal lines drawn through the points of equal division of the circle with the center Ω (ω, ω'). The tangents to the circle and the Oy axis give the apparent lateral outline (fig. 3).

In horizontal plane, the generatrixes are parallel to the Ox axis and they are drawn through the points of division of the director circle resulted from the lateral plane. From the lateral and horizontal projection of the conoid results the vertical projection.

4. GENERATING THE SURFACES USING THE COMPUTER

Using the mathematical formulas from the previous chapter and the Maple software and graphics, we obtained spatial images of the conoid surfaces studied:
- the right conoid from figure 1 appears 3D generated in figure 4;
- the right conoid from figure 2 appears 3D generated in figure 5;
- the right conoid from figure 3 appears 3D generated in figure 6.

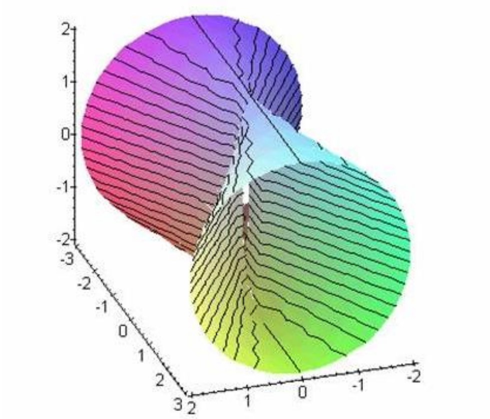

Fig.4 Right circular conoid a = c = 0, b=4, R=2

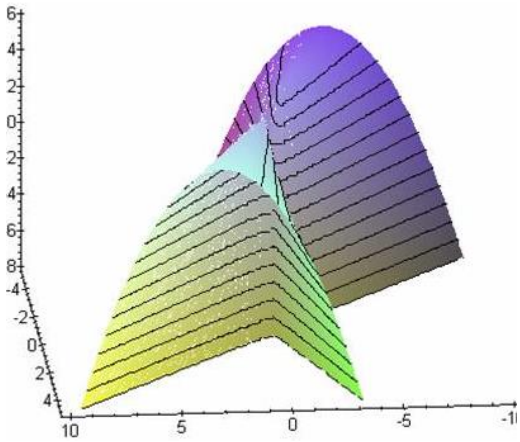

Fig.5 Right conoid with parabolic directrix a=4, b=c=5, p=2

5. PLANE SECTIONS IN CONOIDS

Although the conoid surfaces have a similar configuration with the conical surfaces, their plane sections differ because they are not conical curves.

It has been demonstrated that the intersection with various cutting planes are 4 and sometimes 3 degree curves. The parallel plane with the directrix that cuts the surface by an ellipsis and the parallel plane with the director plane that cuts the surface by two straight-line elements, make the exception.

The sections can be obtained by the three methods employed here and when constructing the surfaces: analytical, descriptive and computer-aided.

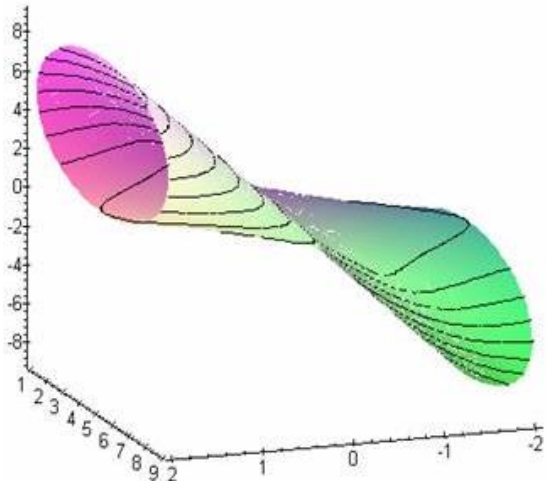

Fig.6 Right circular conoid a=2, b=4, c=4, R=4

We considered the right conoid from [3] with a circular directrix in the horizontal projection plane, axis - a line perpendicular to plane yOz, and the director plane – the lateral projection plane.

Using the method exposed in chapter 2, the equation of the conoid results from the system:

$$\begin{cases} x=\alpha, \quad y=\beta(c-z)+d \\ z=0 \\ (x-a)^2+(y-b)^2-R^2=0 \end{cases} \tag{13}$$

and is:

$$(x-a)^2+\left(c\,\frac{y-d}{c-z}+d-b\right)^2-R^2=0 \tag{14}$$

where **a**, **b**, **c**, and **R** have the significance from figure 7 and **d** is the distance between the line of striction of the conoid and the plane xOz.

Considering the cutting plane parallel with the axis of the conoid, so parallel to Ox and taking b=d, the intersection curve will result from the system:

$$\begin{cases} (x-a)^2+\left(\frac{y-b}{c-z}\right)^2 c^2-R^2=0 \\ z=my+n \end{cases} \tag{15}$$

Its projections on the horizontal plane [H], respectively the vertical one [V] are obtained by eliminating z, respectively y from the equations of the system (15):

$$(x-a)^2+\left(\frac{b-y}{my+n-c}\right)^2 c^2-R^2=0 \tag{16}$$

and:

$$(x-a)^2+\left(\frac{z-n-bm}{c-z}\right)^2 \frac{c^2}{m^2}-R^2=0 \tag{17}$$

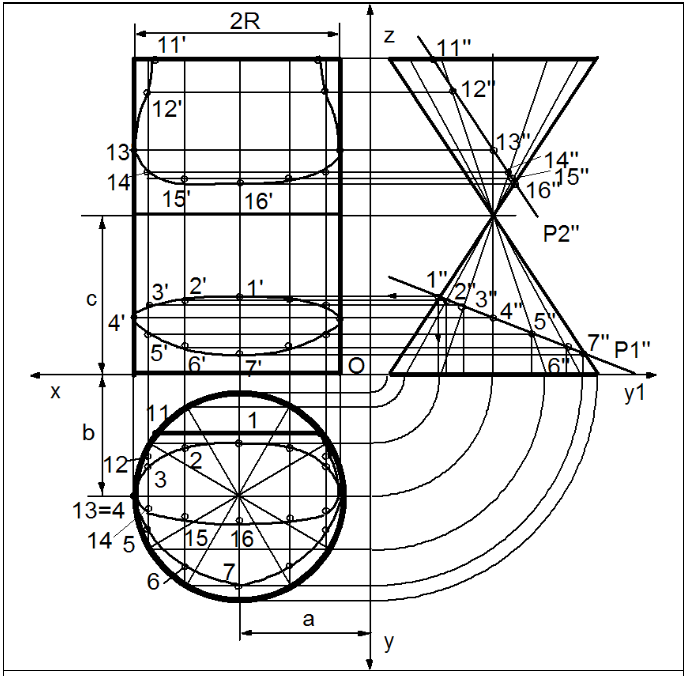

Fig.7 Plane sections in the right conoid (epure)

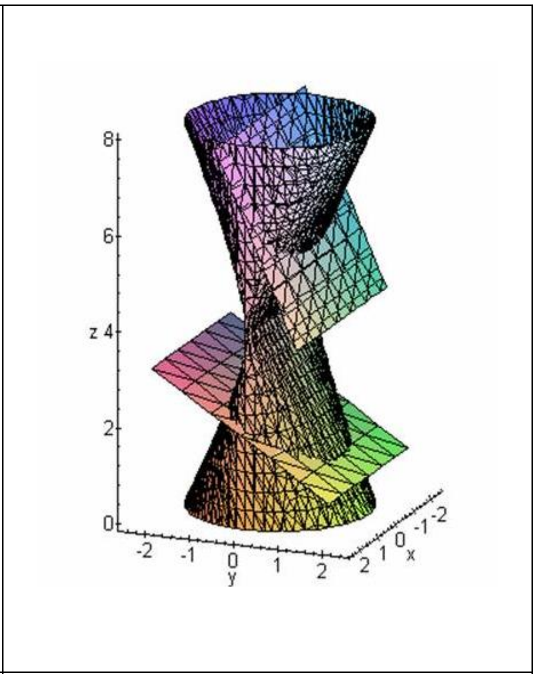

Fig.8 Plane sections in the right conoid (Maple)

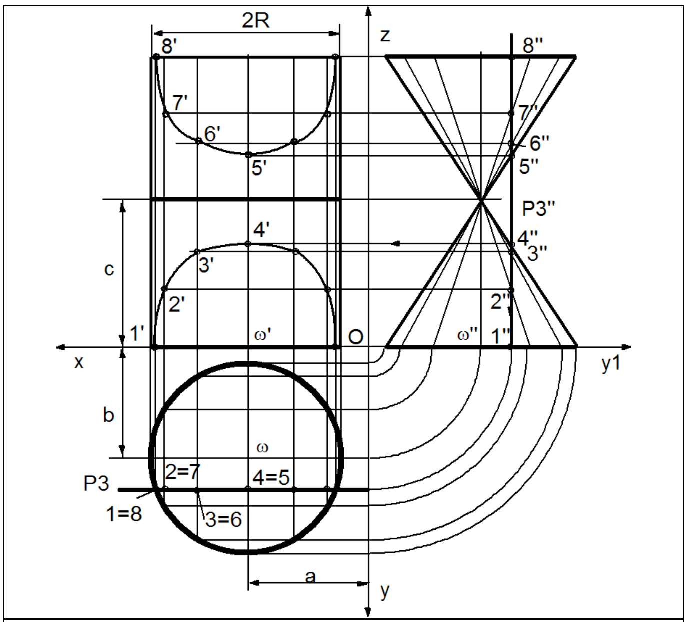

Fig.9 Section with a frontal plane in the right conoid (epure)

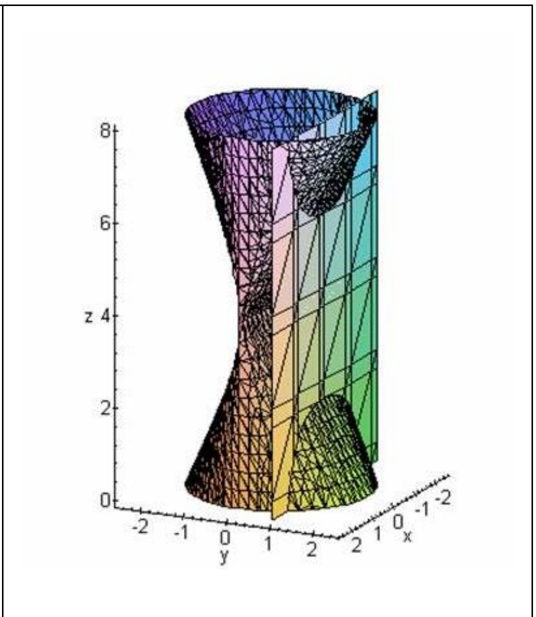

Fig.10 Section with a frontal plane in the right conoid (Maple)

In Fig. 7 and 9 there are the intersection curves using descriptive geometry for the three cases of cutting planes. Plane P1 meets only the inferior nappe of the conoid after the closed curve 1, 2 …7.

Plane P2 that meets the superior nappe and is parallel to the generatrix having x=a, determines the open curve with infinite branches 11, 12…16. The fthird plane P3 cuts both nappes of the conoid by the open curves with infinite branches 1…4 and 5…8.

The same sections are obtained with the help of the Maple software and the formulas (15) in figures 8 and 10 for certain values of the parameters a, b, c, m,n

6. CONCLUSIONS

Conoid surfaces can have a multitude of shapes and therefore multiple applications in constructions and architecture. Of them, the most important ones are: arches, arcades, awnings, roofs, thin envelopes, retaining walls.

As they are so interesting for both the descriptive geometry and the industry, further research in this field is worth doing.

REFERENCES

[1] *E., Murgulescu, N., Donciu, V., Popescu,* Geometrie analitică în spaţiu şi geometrie diferenţială, E.D.P., Bucureşti, 1974.

[2] *M., Orban,* s.a., Geometrie descriptivă. Suprafete şi corpuri cu aplicaţii în tehnică,U.T.Pres, Cluj-Napoca, 2002, pag.13, 37.

[3] *A., Tănăsescu,* Geometrie descriptivă. Perspectivă şi axonometrie, E.D.P., Bucureşti, 1975, pag.267-290.

[4] www.mathcurve.com/surfaces/conoide, R., Ferreol, 2005.^

[5] www.math.ugal.ro/generarea suprafeţelor, M., Aprodu, accesat 2013.

[6] www.wikipedia, conoid, accesat 2013.