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Particular Cases when Constructing a Sphere

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Abstract: This paper solves and analyses a geometry problem using geometry, mathematics and descriptive geometry. The paper ends by result analysis.

Keywords: sphere, tetrahedron, radius, horizontal projection.

1. INTRODUCTION

The paper aims to solve a spatial geometry problem using geometry elements that the students know from high school, as well as descriptive geometry elements.

Let us consider a regular tetrahedron having one edge of size a. The problem wishes to determine the radiuses of the circumscribed sphere and respectively the sphere inscribed in the tetrahedron.

The second chapter of the paper will solve this problem using mathematical geometry.

The third chapter solves the same problem using descriptive geometry elements and for this, we consider the regular tetrahedron with the base in the horizontal projection plane.

Chapter 4 also solves the problem using descriptive geometry but this time we wish to complicate things a little and consider the regulate tetrahedron as having the base in a plane that forms a random angle with the horizontal projection plane.

The paper ends with discussions regarding result analysis.

2. SOLVING THE PROBLEM BY MEANS OF GEOMETRY

A sphere is circumscribed to a tetrahedron if all the vertices of the tetrahedron are on the surface of the sphere, which means all its vertices are at the same distance from the center of the sphere. Therefore, the center of the sphere circumscribed to a tetrahedron is the intersection point of all median planes of the lateral edges and basis of the tetrahedron.

A sphere is inscribed into a tetrahedron if it is tangent to all the faces of that tetrahedron (sides and base). In other words, the center of the sphere is inscribed in the tetrahedron is equidistant from all the sides of the tetrahedron, so the center of the sphere is the intersection point of the bisectrix planes of all dihedral angles of the tetrahedron.

Figure 1 presents the regular tetrahedron VABC.

We mark with *O* the center of the sphere that is circumscribed to this tetrahedron, with O_1 we mark the center of the circle that contains the ABC base, and with *V'* we mark the symmetrical point of *V* in relation to O. So *VV'* is the diameter of the sphere.

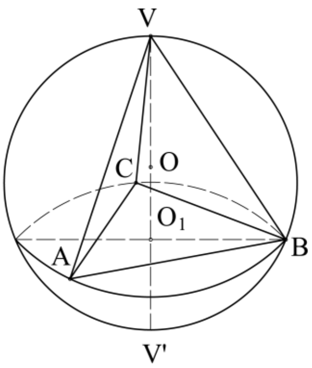

Fig. 1. The regular tetrahedron VABC

The O_1 center of the circle circumscribed to the equilateral triangle ABC, with side a that coincides with the center of gravity of the triangle.

Within the ABC triangle we draw the height BE. Within the BEC triangle, we can apply the Pythagorean Theory, thus resulting:

$$BE^2+EC^2=BC^2 \quad (1)$$

The median line from the vertex B is market with m_B. Results:

$$m_B = \sqrt{a^2 - \left(\frac{a}{2}\right)^2} \quad (2)$$

Therefore:

$$BO_1 = \frac{2}{3} m_B = \frac{2}{3} \cdot \frac{a\sqrt{3}}{2} = \frac{a\sqrt{3}}{3} \quad (3)$$

Within the right angle triangle VO_1B se we apply the Pythagorean Theory, thus resulting:

$$VO_1 = \sqrt{VB^2 - O_1B^2} = \frac{a\sqrt{6}}{3} \quad (4)$$

Within the right angle triangle V*BV'* (where *VV'* is the diameter of the sphere, plane *VBV'* will section the sphere in a larger circle that contains the triangle *VBV'*) we apply the cathetus theorem:

$$VB^2 = VV' \cdot VO_1 \quad (5)$$

The relation (5) can also be written as:

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$$a^2 = 2R\frac{a\sqrt{6}}{3} \qquad (6)$$

hence the radius of the sphere:

$$R = \frac{a\sqrt{6}}{4} \qquad (7)$$

In order to determine the radius of the sphere inscribed in the regular tetrahedron we will decompose the VABC tetrahedron into four tetrahedron with the vertex in I, the center of the inscribed sphere and the sides of the tetrahedron. The distance from I to each side is the very radius r, the radius of the sphere inscribed in the tetrahedron.

The volume of the tetrahedron v will be written as the sum of the four tetrahedron. Thus:

$$v = 4\frac{r \cdot S}{3} \qquad (8)$$

where S is the surface of one side. However:

$$S = \frac{a^2\sqrt{3}}{4} \qquad (9)$$

results that the volume of the sphere will be:

$$v = \frac{1}{3}\cdot\frac{a^2\sqrt{3}}{4}\cdot\frac{a\sqrt{6}}{3} = \frac{a^3\sqrt{2}}{12} \qquad (10)$$

From the relations (8), (9) and (10) results that:

$$r = \frac{3v}{4S} = \frac{a\sqrt{6}}{12} \qquad (11)$$

The sphere stands on the edges of the tetrahedron in the tangent points. The segments that leave from a vertex of the tetrahedron to the tangent points are equal (tangent to one point exterior to the sphere). Since the tetrahedron is a regular one, all these tangents from the four vertices are equal, which shows that the tangent points coincide with the middle of the sides of the tetrahedron.

Let us take the *M*, *N*, *P*, *Q* , middle (of two pairs of opposed edges in the tetrahedron) of the edges [AB], [CV], [AC] and [BV] (Fig. 2).

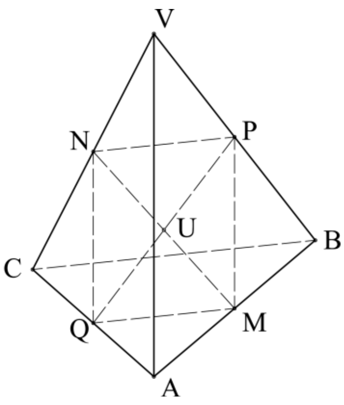

Fig. 2. The tetrahedron

The quadrilateral MNPQ is a square. The sides MQ, MP, PN, NQ are middle lines:

$$MQ = MP = PN = NQ = \frac{a}{2} \qquad (12)$$

In the BQV and MVC congruent triangles, the segments QP and MN are median lines. Thus, we have:

$$QP = MN \qquad (13)$$

From the UPM right angle triangle results:

$$2UP^2 = \frac{a^2}{4} \qquad (14)$$

hence:

$$UP = \frac{a\sqrt{2}}{4} \qquad (15)$$

which represents the radius of the requested sphere.

3 FIRST CASE OF SOLVING THE PROBLEM USING DESCRIPTIVE GEOMETRY

We take the regular tetrahedron with the base situated in the horizontal projection plane (fig. 3). The side of the tetrahedron is marked with a.

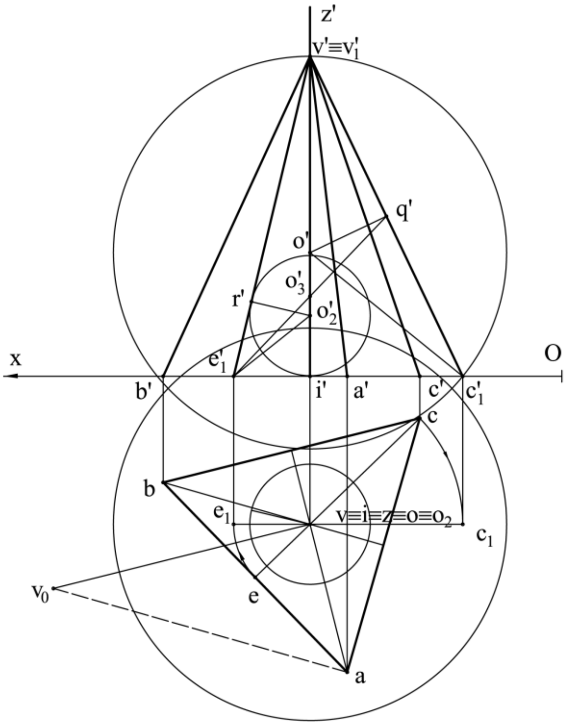

Fig. 3. Constructing a sphere

In the case of the VABC tetrahedron with the base in the horizontal projection plane, the ABC base is displayed in its real size. The abc horizontal projection plane being an equilateral triangle, the vertex of the tetrahedron is projected in the center of gravity of the triangle. The foot of the height I(i, i') of the tetrahedron is also projected in the same point. In vertical projection, *a'*, *b'* and *c'* are projected on the land line since they value zero (the tetrahedron has the base on the horizontal projection base).

The problem is to determine the height of a regular tetrahedron that has side a. In this respect, the horizontal projection builds the right angle triangle avv_o that has a cathetus the *va* projection and the hypotenuse the length a of the edges. In this triangle, the cathetus is vv_o the height of the tetrahedron and respectively the value of the vertex S. The foot of the height of the tetrahedron is a point situated in the horizontal projection plane. Its value is zero. So, i' is on the ground line. We measure the value of point S that gives the vertex of the tetrahedron. By joining the projection v' with projections a', b' and c', we get the vertical projection of the tetrahedron.

Next we construct the sphere that is circumscribed to the tetrahedron. Since the tetrahedron is a regular one and it has its base on the horizontal projection plane, the *v* projection will also project *o*, the center of the circle circumscribed to the tetrahedron. Since the base of the tetrahedron is on the horizontal projection plane (where it appears in it real size), the radius of the circle circumscribed in horizontal projection is equal to *va*, respectively *vb*, respectively *vc*.

We are interested next to determine the vertical projection of this point.

Through the vertex of the tetrahedron we construct a vertical rotation axis. Around this axis, we rotate the *vc* edge until it is in frontal position. In vertical position we find the real size of the edge, that is the $v_1'c_1'$ projection. The right angle triangle $v'i'c_1'$ is formed in vertical projection. In this triangle, the median line of the side $v_1'c_1'$ is constructed. Since in vertical position $v_1'c_1'$ appears in real size, the projection q' of the middle of the side is found and a perpendicular line is drawn. The point where the perpendicular line meets the projection $s'i'$ we can find the vertical projection o' of the center of the sphere circumscribed to the tetrahedron. The $v'o'$ and $o'c_1'$projections represent the radius of the sphere circumscribed to the tetrahedron and it appears in real size.

Next we find the vertical projection of the center of the sphere inscribed in the tetrahedron. We mark with *e* the foot of the perpendicular line constructed from the projection *c* on the ab side. After the level rotation performed before, we bring e in the position of e_1. We thus obtain a frontal line that appears in real size in vertical projection, that is in projection $v_1'e_1'$.

The point where this bisector line intersects the projection $v_1'i'$ finds the vertical projection $o_2'$of the sphere inscribed in the tetrahedron. If from the projection o_2' we build a perpendicular on the projection $v_1'e_1'$ we obtain $o_2'r'$. The projection $o_2'r'$ represents the radius of the sphere inscribed in the tetrahedron. This projection appears in real size.

The center of the sphere tangent to all the edges of the tetrahedron are in the middle of the projection $e_1'q'$, that is the projection o_3'. The real size of this sphere is the projection $o_3'\,e_1'$, respectively $o_3'\,q'$.

4. THE SECOND CASE OF SOLVING THE PROBLEM USING DESCRIPTIVE GEOMETRY

In this case we consider the regular tetrahedron with the base in a different plane than the projection planes. We know the projections of the radius of the circle circumscribed to the base and the angle, which the base makes with the horizontal projection plane. The situation considers the angle that the base makes with the horizontal projection plane as being 45° (Fig.4).

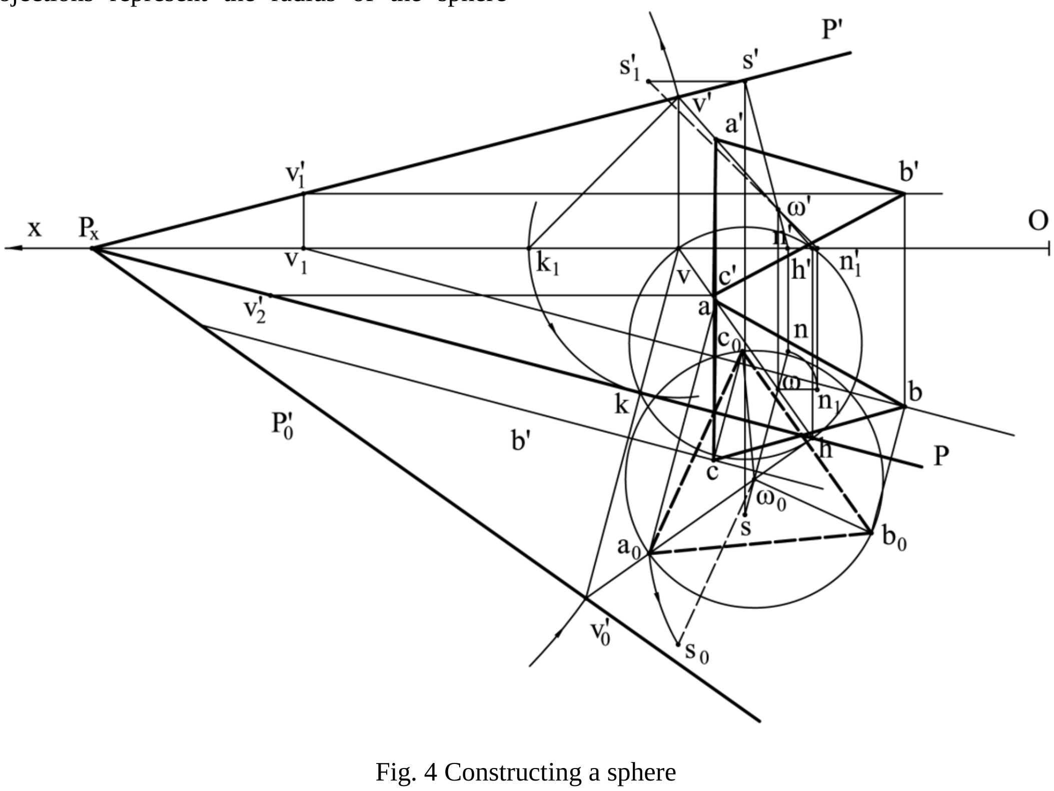

Fig. 4 Constructing a sphere

The problem represents the radius of the circle circumscribed to the base in both horizontal and vertical projection, that is ao_1 and $a'o_1'$. Next we construct plan P that passes through the line (ao_1 , $a'o_1'$) at an angle of 45° with the horizontal projection plane.

In order to construct plan P, we determine the traces of the line (ao_1 , $a'o_1'$) and we construct the projection $v'i_1$ at an angle of 45° from the OX axis. The arc radius vi_1 is drawn until it meets the circle with a hv diameter in *i*. The horizontal of the plane is *hi*, and the vertical one results immediately.

Plane P is folded on the horizontal projection plane with the help of the vertical line. The projections a and o1 are also folded. In this fold, we represent the circle with the radius $a_o o_{1o}$. In this circle, the equilateral triangle $a_o b_o c_o$ is inscribed. This triangle is folded and turned thus obtaining the horizontal and vertical projections of the base triangle.

Next we need to determine the height of the tetrahedron. This is determined as it has been explained in the previous paragraph, resulting projection v_0. The true size of the height of the tetrahedron is determined that we place in $o_1'v_1'$ on the frontal rotated position of the perpendicular (o_1n, $o_1'n'$) drawn in (o_1, o_1') on plan P. Returning from the rotation, we obtain V(v,v'), the vertex of the tetrahedron.

Determining the center, respectively the radius of the sphere circumscribed to the tetrahedron is done as in the previous paragraph. This is determined by folding after which the initial position can be assumed.

The problem continues with the construction in the previous paragraph. This is why we gave it up in order not to burden the graphic construction too much.

4 .CONCLUSIONS

This paper analyzed a geometry problem that has been solved using mathematical geometry and descriptive geometry.

By solving the problem with the help of mathematical geometry, we can determine some elementary notions, such as the radius of the three spheres and their centers.

By solving the same problem with the help of descriptive geometry, in the case of a regular tetrahedron with the base in the horizontal projection plane, the result is an intuitive construction. The centers of the three spheres and their radiuses can be more quickly determined and simpler than in the previous case.

A more difficult construction results in the case of the regular tetrahedron situated in a random plane.

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