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Considerations of Curvature Determination of Some Specific Plane Curves

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Abstract: The plane curves and, among them, the conical sections resulting at the intersection of some surfaces with planes in given conditions, are frequently encountered in practice as components of geometrical configurations of some mechanical parts. This paper is aiming at tackling ways of graphically determining the curvature centers of conical sections and their evolutes, as well as of geometrical properties of theirs, which makes possible their utilisation in the construction of some complex solids composed of combinations of cylindrical and conical surfaces.

Keywords: plane curves, center of curvature

1. INTRODUCTION

Among the plane curves, the conical sections are exceptionally important, as they result at intersections between a conical surface and any plane, perpendicular to one of the projection planes or at the intersection of two second degree surfaces [1]. They are curves having variable curvatures. Their actual size is graphically determined through one of the known methods of transformation of projections [2]. In some practical cases, it is useful to approximate an arc of the conical curve through an arc of circle having the same curvature as the given curve in the respective point. The graphical construction of the curvature center of a curve is based upon the fact that if a mobile line remains, along its movement, normal to a given curve, the point in which it intersects its involute is the curvature center, and the curvature center of the given curve evolute.

Within the paper, two conical curves, frequently encountered in the geometrical configurations of machine parts, were taken into consideration, namely the ellipse and the parabola.

1.1. Determining the curvature center of ellipse

If the mobile line stays normal to the ellipse in any of its points, the ellipse and its axes divide the line into proportional segments [3]. Considering point *M* from the ellipse (Fig. 1), tangent *ut*, the normal *Mv* and the axes of the ellipse are four lines forming similar triangles, i.e. triangle *FMF'* and *uMv*. The circumscribed circles to these triangles intersect in point *f* lying at the intersection between *vt* and *ua*, *a* is the point in which the normal line intersects the big axis of the ellipse. The circle passing through point *f* and tangent in *M* to *ut* meets the normal line in point *c*, in which this line intersects its involute, that is, in the curvature center of the ellipse. Hence, it comes that the curvature center *c* is the intersection point between the normal line drawn in *M* and the perpendicular line *fc* to *fM*.

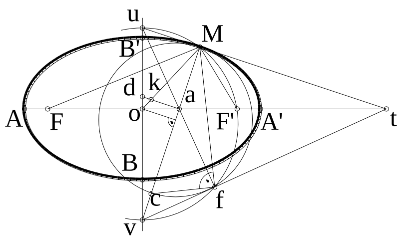

Fig. 1. Determining curvature center of the ellipse [3].

The relations settled in [3] between the sides of the formed similar triangles show that the points *c* and *M* divide the segments *Mv* and *ut*, respectively, in the same proportion, that is: $\frac{tM}{Mu} = \frac{ac}{cv} = \frac{ak}{kd}$. Based on these, some constructive variants results for the curvature center in a point *M* of the ellipse [3]:

a) In the point where the normal to ellipse in point *M* intersects one of the axes, the perpendicular is drown to it (Fig. 2 a). The perpendicular intersects the diameter passing through *M* in point *k*. The perpendicular drawn from *k* onto the big axis of the ellipse intersects the normal line in the curvature center *c*, required;

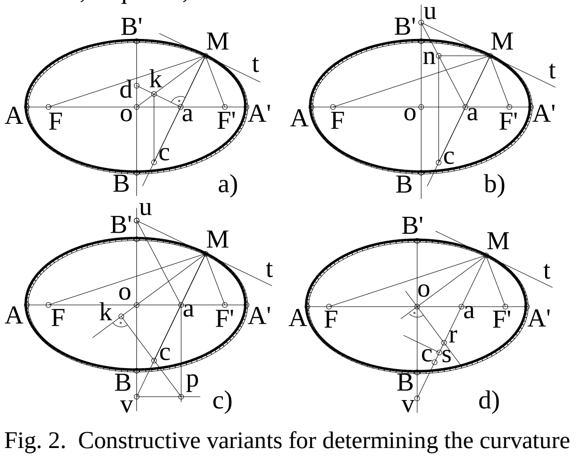

Fig. 2. Constructive variants for determining the curvature center of an ellipse.

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b) The parallel to the big axis *AA'* of the ellipse drawn from *M*, intersects line *ua*, in point *n* (Fig. 2 b). The parallel to the small axis *BB'* of the ellipse, drawn from point *n*, cuts the normal corresponding to *M*, in the required curvature center, *c*;

c) The perpendiculars drawn from points *a* and *v* onto the axes of the ellipse, intersect in point *p*, and the perpendicular drawn from *p* onto the diameter *oM*, intersects the corresponding normal *M*, in the curvature center *c* (Fig. 2 c);

d) The perpendicular drawn from *o* onto *oM* meets the normal *Ma* in point *r*, and its symmetric point with respect to the middle of *s*, *as* = *sv*, is the curvature center *c* (Fig. 2 d).

1.2. Determining curvature centre of parabola

a) In the case of parabola, to graphically determine curvature center in a point *M* [3], a parallel is, firstly, drawn from point *M* to its axis (Fig. 3). This line intersects the perpendicular on the normal line drawn from *n*, in point *g*. The perpendicular drawn from *g* onto the parabola axis intersects the normal line in the curvature center *c*.

The line *ng* intersects the vector radius *Mf* in point *e*, and line *ec* is the perpendicular on *Me* since *Mn* is the bisector of angle *gMe*. One observes that *Mf* = *Fe* and, hence, it comes that the perpendicular on line *me* drawn from point *e* intersects the normal in the curvature center *c*. Thus, it results that the curvature radius of parabola corresponding to a point *M* on the curve is the double of the segment of line contained on the normal in *M*, between point *M* and the point in which the normal line intersects the director line, that is *cM = 2 Mr*.

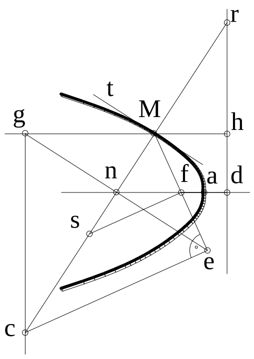

Fig. 3. Determining the curvature center of the parabola [3].

2. RESULTS AND DISSCUTIONS

2.1. The conical section is an ellipse

According to Dandelin's theorem [4], the elliptical section in the cone results when a parallel plane to a secant plane drawn through the cone vertex intersects two of its imaginary generatrices.

The paper analyses the cases where the secant plane forms angles of 45^0 respectively 60^0 with the horizontal projection plane and the diameter of cone base is d =54 mm (Fig. 4).

The plane sections effected in the cone are in both cases ellipses. The actual sizes of the resulted

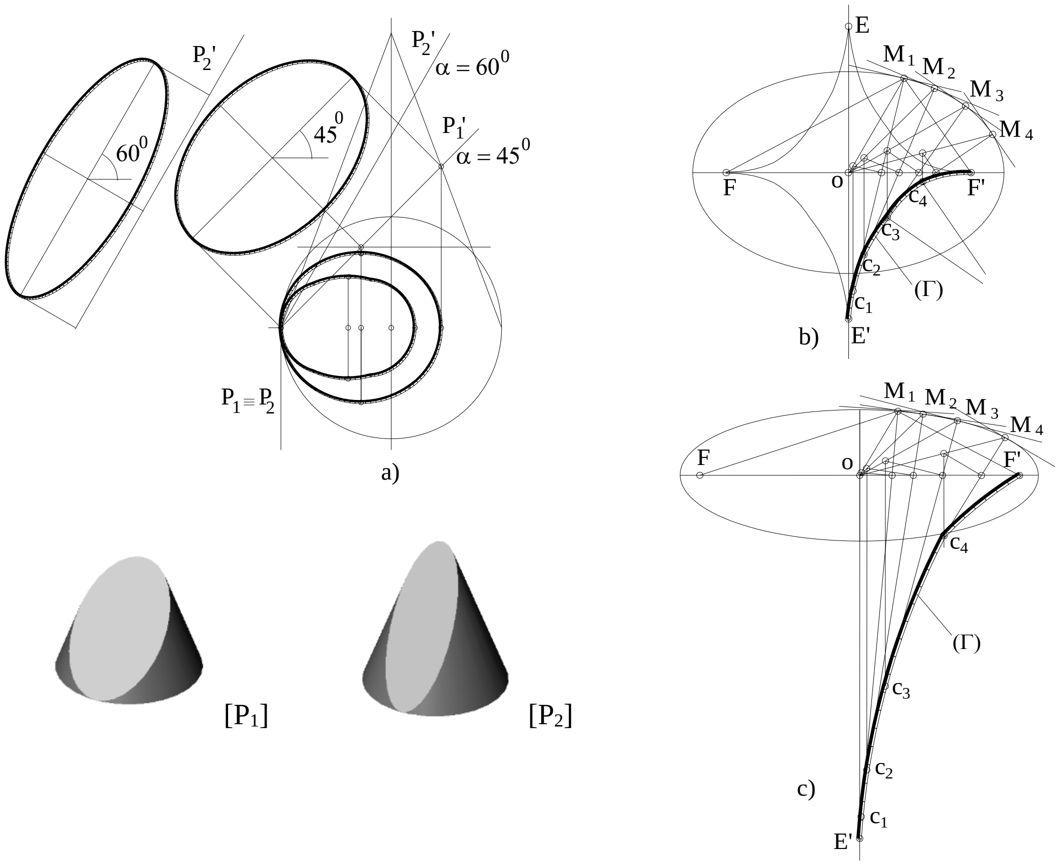

Fig. 4. Determining the curvature centers of ellipses and astroid – the ellipses involutes.

ellipses are obtained by revolving their plane on the vertical
projection plane.

Knowing the axes of ellipses, $2a$ and respectively $2b$, has been established [5]: the focal distance $c = \sqrt{a^2 - b^2}$, excentricity $e = c/a$, the curvature radius R calculated in the considered point M, the curvature in point M, $k = 1/R$ and the length of the ellipse. The graphical determination of curvature centers is presented in Figure 4, and the calculated values are given in Table 1, the linear dimensions being expressed in [mm]. For the practical realisation of some parts, it is important the length of the ellipse resulted at the intersection between the plane [P] and the considered cone. For its calculation, the approximate formula was used [5], which provides the required precision in practice and not the exact formula, which, being a curvilinear integral, presents difficulties in solving. On each ellipse, there were considered four points, corresponding to angles of 15^0, 30^0, 45^0 and 60^0 formed by the vector radius with Ox axis.

Table 1. The determined characteristics of the ellipses.

d=54	a	e=c/a	M	R	K= 1/R	L
[P_1] 45^0	a=27,7 b=18,1 c=21,0	0,8	15^0	15,1	0,06	145,6
			30^0	25,7	0,04	
			45^0	32,4	0,03	
			60^0	39,9	0,02	
[P_2] 60^0	a=32,7 b=12,3 c=30,3	0,9	15^0	21,6	0,04	149,0
			30^0	51,7	0,01	
			45^0	67,8	0,01	
			60^0	76,0	0,01	

In Figure 5 is presented the variation of ellipse curvature, determined in the two cases, as function of point M position on the ellipse and inclination of the secant plane with respect to horizontal projection plane.

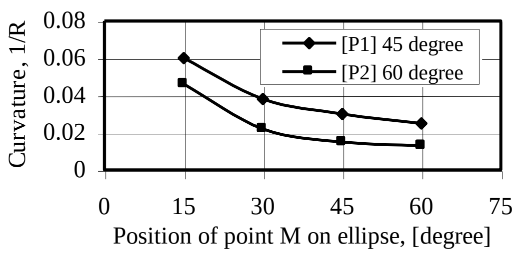

Fig. 5. The variation of the considered ellipse curvature as a function of point M position.

The points F and F' correspond to the minimum curvature radius $AF=A'F'$ and vertices E and E', to the maximum curvature radius, $B'E = BE'$.

The normal lines to ellipses envelope curves [Γ], which are elongated astroids and represent the involutes of ellipses obtained as sections in the same revolution cone. The curvature centers c_1, …c_4, previously determined (Fig. 4), are points of the astroids. In both cases, the axes of astroids are superposed on the ellipse axes, and their vertices are return points. The normal lines to ellipses are tangents to the astroid and normal to the astroid envelope its involute.

Knowing that the normal line to the ellipse is divided by the ellipse and its axes into proportional segments, its curvature centers envelope another curve, whose curvature centers lie on the ellipse involute. For their determination [3], one has to be used the construction of curvature centers for the astroid resulted from the intersection of cone with plane [P_2], studied above (Fig. 6).

Knowing for example, the curvature center c_2 of the ellipse, one draws from a the perpendicular onto Ox and measures $aa' = a'a''$. Similarly, a parallel through e is drawn to Ox and $ee' = e'e''$ is measured. The perpendicular $a''p'$ and $e''h$ are drawn which intersect in point h'. The parallel drawn through e' to ae, determines, on the perpendicular to the normal line drawn through c_2, the curvature center c_2' of ellipse involute corresponding to point M_2 of the ellipse. The result is in conformity with [3], that is $c_2c_2'=3c_2p$, proving the correctitude of construction, more simple that the classical ones. Point c_2' is directly obtained, measuring from point c_2, on the perpendicular to the normal M_2c_2, a segment, three times the length of c_2p measured on this perpendicular, contained between the diameter M_2O of ellipse and the normal line M_2c_2.

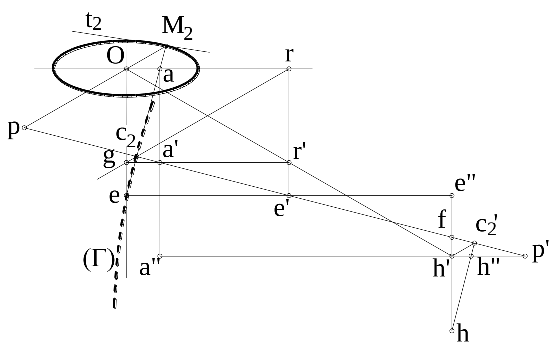

Fig. 6. Determining the curvature center of the studied ellipse corresponding to point M_2.

2.2. The conical section is a parabola

The parabolic section in the cone develops in the case when the plane drawn through the cone vertex, parallel to the secant plane, intersects the cone along two confounded generatrices [4]. The secant plane [P_1] has, in this case, the vertical trace P_1' parallel to the cone generatrix (Fig. 7). Determination of parabola was made by methods known in descriptive geometry, and its actual size, resulted by revolving of the plane [P_1], which contains it, on the horizontal projection plane. For determining the curvature centers, the method presented in [3] has been used. As the current point M approaches to the parabola's vertex, the curvature radius diminishes.
The curve enveloped by the normals to the parabola, respectively the locus of curvature centers is curve [Γ], having the same axis as the given parabola, while its vertex, C, is a return point. Point N is the point in

which the parabola and its involute intersect. If from N a perpendicular is drawn onto axis Ox and then $PR = FD$ is measured, the segment NR is normal in N to the parabola.

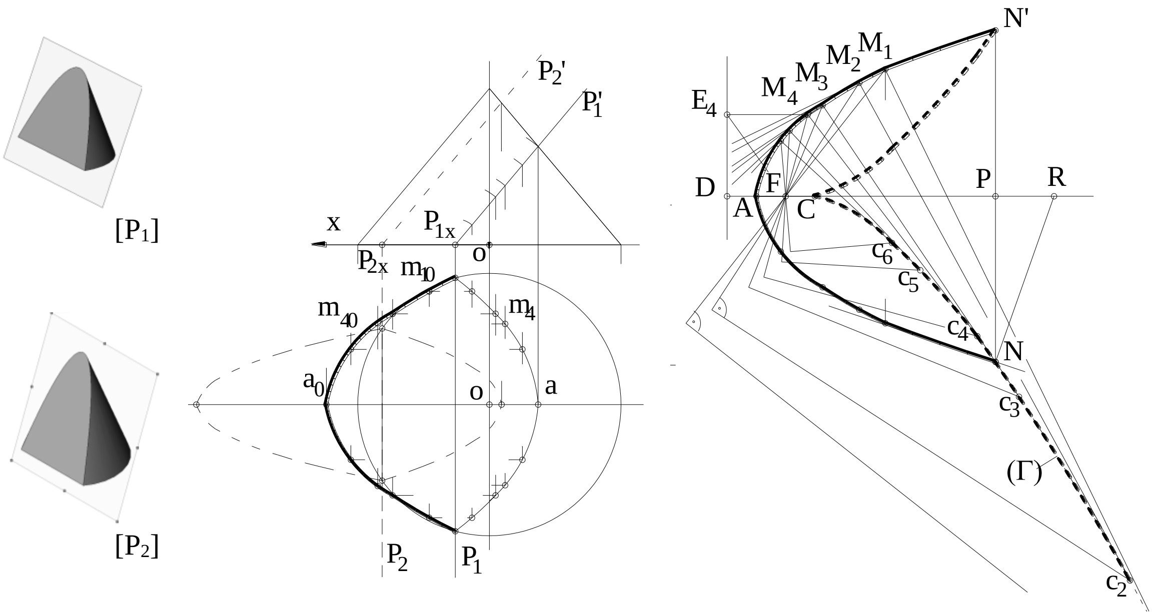

Fig. 7. Determining curvature centers of parabola and parabola involute with [P_1].

In Fig. 7 is also represented the parabola obtained by the intersection of the cone with plane [P_2] for which $OP_{2x} > OP_{1x}$.

Determination method of the curvature centers, in case when plane [P_2] is used, is similar to that applied in the case of the parabola obtained with plane [P_1]. One has to be observed that as points P_x of the secant planes get further from center o' of the base, the curvature radii taken in various points of parabola diminish towards the parabola vertex.

The parabola involute is a semicubic parabola having a reversal point C, this being the symmetric of vertex A of parabola with respect to focus F [6]. The construction of curvature center of the parabola involute (Fig. 8), results knowing that the parabola divides in constant proportion the segments contained on normal lines between the involute and the directrix of parabola respectively [3].

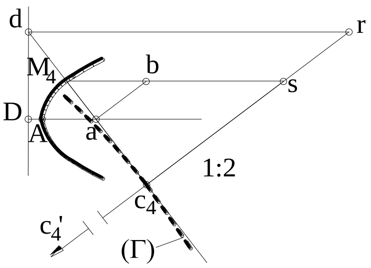

Fig. 8. Determining the curvature center of the studied parabola evolute, corresponding to point M_4.

Point c_2' being the curvature center of involutes (Γ) and r the intersection point of perpendicular dr on the directrix and normal M_4c_4' to the involute, it should be that $c_4c_4' = 2c_4r$. Drawing the diameter M_4s of the parabola, one finds $c_4s = 2\ c_4r/3$, as $c_4M_4 = 2c_4d/3$ and thus results $c_4c_4' = 3c_4s$.

3. CONCLUSIONS

The conical curves resulted as plane sections in the cone or at the intersection of two second degree surfaces, are drawn through points depending on the relative position of geometric elements already presented, one with respect to others or with respect to the projection planes, and function of these, they may be ellipses, parabolas or hyperbolas. Out of these, the paper has tackled the graphical determination of curvature and its variation for ellipse and parabola, as parts of geometrical configurations of some machine parts.

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